GS/ECON 5010 Answers to Assignment 3 November 2010

Q1. Suppose that there are two states of the world, and that the probability of state #1 is π (with $0 < \pi < 1$). An investor is choosing how to allocate her wealth among three assets.

Asset a pays a net return of 5% in state #1, and 20% in state #2.

Asset b pays a net return of 10% in state #1, and 10% in state #2.

Asset c pays a net return of 0% in state #1, and 40% in state #2.

Investors are not allowed to go short. (That is, they must hold non–negative quantities of each asset.) Which assets might be held in the portfolio of a risk–averse utility maximizer?

Explain briefly.

A1. Note that the question did **not** give any information about the probability of either state of the world.

So suppose that the probability of state #1 were very high : 0.8 for example. Then any risk–averse investor would want to put all her money in asset b, since it is a sure thing, and offers a higher expected return than either of the other assets.

Now suppose that the probability of state #2 were very high : greater than 0.5 for example. Then asset c offers a higher expected return than either other asset. An investor who was not very risk averse would certainly want to put some of her money into asset c. (In fact any risk–averse utility maximizer would want to put some of her wealth into asset c. If she chooses the quantities A and C to put into assets a and c so as to maximize her expected utility, with the rest of her wealth going into asset b, then expected utility must be increasing with C at $C = 0$.)

That leaves asset a. It turns out that a risk–averse expected–utility–maximizing investor will **never** want to put any of her wealth in asset a. This can be seen from the first–order conditions from her expected utility maximization problem.

But more directly, asset a is **dominated** by a combination of assets b and c. Suppose that she invests \$1 less in asset a, and instead puts β dollars in asset b and $1 - \beta$ dollars in asset c. Her return in state #1 will increase from this move if (and only if)

$$10\beta > 5$$

or

$$\beta > 0.5 \tag{1 – 1}$$

Her return will increase in state #2 if (and only if)

$$10\beta + 40(1 - \beta) > 20$$

or

$$\beta < \frac{2}{3} \tag{1 – 2}$$

Now there are values of β which satisfy **both** equations $(1-1)$ and $(1-2)$: $\beta = 0.6$. This means that she can increase her return in each state of the world, simply by replacing each dollar invested in asset a with 60 cents in asset b and 40 cents in asset c.

That is, asset a is stochastically dominated by a convex combination of assets b and c.

So, no matter what her preferences, and no matter what the probability π of state #1, she should never invest any of her wealth in asset a.

Q2. How much would a risk–averse utility maximizer invest in asset c, if her utility–of–wealth function were

$$u(W) = \frac{1}{1-\beta} W^{1-\beta} \qquad \beta > 0$$

and the asset returns are as described in question 1?

A2. From the answer to question 1, she should never put any of her wealth in asset a.

So her problem is to choose an amount X to put in asset c, with the remainder of her wealth going into asset b. Her expected utility will be $\pi U((1.10)(W-X) + (1-\pi)U((1.10)W + (0.3)X)$. Given her preferences, she should therefore choose X to maximize

$$\pi \frac{1}{1-\beta} W^{1-\beta} [1 + (0.1)(1-x)]^{1-\beta} + (1-\pi) \frac{1}{1-\beta} W^{1-\beta} [1.10 + 0.3x]^{1-\beta} \tag{2.1}$$

if $x \equiv \frac{X}{W}$ is the proportion of her wealth that she invests in asset c.

The first–order condition for maximization of $(2-1)$ with respect to x is

$$-\pi(0.1)[1 + (0.1)(1-x)]^{-\beta} + (1-\pi)(0.3)[1.1 + 0.3x]^{-\beta} = 0 \tag{2 − 2}$$

or

$$\pi^{-1/\beta}[1 + (0.1)(1-x)] = (1-\pi)^{-1/\beta} 3^{-1/\beta} (1.1 + 0.3x) \tag{2 − 3}$$

so that

$$\pi^{-\frac{1}{\beta}}(1.1 - 0.1x) = [3(1-\pi)]^{-\frac{1}{\beta}}(1.1 + 0.3x) \tag{2 − 4}$$

or

$$\pi^{-\frac{1}{\beta}}(11 - x) = [3(1-\pi)]^{-\frac{1}{\beta}}(11 + 3x) \tag{2 − 5}$$

which is equivalent to

$$11\pi^{-\frac{1}{\beta}} - x\pi^{-\frac{1}{\beta}} = 11[3(1-\pi)]^{-\frac{1}{\beta}} + 3x[3(1-\pi)]^{-\frac{1}{\beta}} \tag{2 − 6}$$

or

$$11\pi^{-\frac{1}{\beta}} - 11[3(1-\pi)]^{-\frac{1}{\beta}} = x\pi^{-\frac{1}{\beta}} + 3x[3(1-\pi)]^{-\frac{1}{\beta}} \tag{2 − 7}$$

Solving $(2-7)$ for x,

$$x = \frac{11\pi^{-\frac{1}{\beta}} - 11[3(1-\pi)]^{-\frac{1}{\beta}}}{\pi^{-\frac{1}{\beta}} + 3[3(1-\pi)]^{-\frac{1}{\beta}}} \tag{2-8}$$

The numerator of expression $(2-8)$ is non–negative if and only if

$$\pi \leq 3(1-\pi) \tag{2-9}$$

or

$$\pi \leq \frac{3}{4}$$

So the person chooses to put a non–negative fraction of her wealth in a risky asset only if the probability of the good state #2 is at least 25%. (This cut–off probability does not depend on the value β of the person's coefficient of relative risk aversion : she's willing to invest at least a small fraction of her wealth in the risky asset if and only if the risky asset has a higher expected return than the safe asset.)

If π is small enough, then the value of x satisfying $(2-8)$ exceeds 1, so that the person will want to put all her wealth in the risky asset (c). This threshold value of π, for which the right side of $(2-8)$ equals 1, is

$$\pi = \frac{\frac{5}{7}^{\beta}3}{1+3\frac{5}{7}^{\beta}} \tag{2-10}$$

[Alternatively, $(2-3)$ can be written

$$11 - x = K(11+3x) \tag{2-11}$$

if we set

$$K \equiv [\frac{\pi}{3(1-\pi)}]^{1/\beta} \tag{2-12}$$

So the proportion of the person's wealth going into asset c is

$$x = \frac{1-K}{1+3K}11 \tag{2-13}$$

If $K < 5/7$, expression $(2-13)$ exceeds 1, and the person will want to invest all her wealth in asset c. If $K > 1$, expression $(2-13)$ is negative, and the person will want to put all of her wealth in asset b.

Not surprisingly, the definition $(2-12)$ of K shows that it increases with the probability π of the bad state, and increases with the person's index of relative risk aversion β (when $K < 1$), so that increases in π or in β lead her to put less of her wealth in the risky asset.]

Q3. Is the production function

$$F(x_1, x_2, x_3) = (\sqrt{x_1} + 2\sqrt{x_2} + 3\sqrt{x_3})^3$$

weakly separable? Strongly separable? Explain briefly.

What is the elasticity of substitution for this production function?

A3. This is a CES production function (with the difference from the text, that the "share parameters" on the inputs are not equal to each other).

That means that it is strongly and weakly separable : the marginal rate of technical substitution between inputs i and j is

$$MRTS_{ij} = \frac{i}{j}\frac{\sqrt{x_j}}{\sqrt{x_i}} \tag{3-1}$$

which is independent of the quantity of the third input.

The general formula for the elasticity of substitution between any 2 inputs is that

$$\sigma = \frac{1}{1-\rho}$$

when

$$f(\mathbf{x}) \equiv [\sum_{j=1}^{n}(x_j)^{\rho}]^{\frac{\mu}{\rho}}$$

Here $\rho = 0.5$, so that the elasticity of substitution is

$$\sigma = 2$$

This can be seen directly from the expression $(3-1)$ for the MRTS ; the condition that

$$MRTS_{ij} = \frac{w_i}{w_j}$$

and equation $(3-1)$ imply that

$$\frac{i}{j}\frac{x_j}{x_i}^{0.5} = \frac{w_i}{w_j} \tag{3-2}$$

or

$$\frac{x_i}{x_j} = \frac{w_i}{w_j}^{-2}(\frac{i}{j})^2 \tag{3-3}$$

which means that the elasticity of x_i/x_j with respect to w_i/w_j is -2.

Q4. If a production function $f(x_1, x_2)$ has the equation

$$f(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 + 1)$$

calculate the marginal product of each input, and the marginal rate of technical substitution.

Does the production function exhibit decreasing, constant, or increasing returns to scale? Explain briefly.

A4. Since

$$MP_1 = \frac{\partial f}{\partial x_1} = \frac{1}{x_1 + 1} \tag{4-1}$$

$$MP_2 = \frac{\partial f}{\partial x_2} = \frac{1}{x_2 + 1} \qquad (4-2)$$

the marginal rate of technical substitution is

$$MRTS_{12} = \frac{MP_1}{MP_2} = \frac{1 + x_2}{1 + x_1} \qquad (4-3)$$

Now for the more difficult part, the returns to scale. If we multiply both input quantities by some constant $a > 1$, then

$$f(ax_1, ax_2) = \ln(ax_1 + 1) + \ln(ax_2 + 1) \qquad (4-4)$$

so that the production function exhibits increasing returns if $\ln(ax + 1) \geq a \ln(x + 1)$, for any $x > 0$ and $a > 1$.

So to check returns to scale, one way is to look at the function $\ln(x + 1)$.

THEOREM : $\ln(ax + 1) < a \ln(x + 1)$, for any $x > 0$ and $a > 1$

PROOF : Let

$$G(a) = a \ln(x + 1) - \ln(ax + 1)$$

It must be shown that $G(a) > 0$ when $a > 1$ (because $G(a) > 0$ is the same as $\ln(ax + 1) < a \ln(x + 1)$).

At $a = 1$, $G(a) = 0$.

Treating $G(\cdot)$ as a function of a,

$$G'(a) = \ln(x + 1) - \frac{x}{ax + 1} \qquad (4-5)$$

If $G'(a) \geq 1$ at $a = 1$, then we are done, since $G'' > 0$.

So the problem has been reduced to showing that $G'(1) = \ln(x + 1) - \frac{x}{x+1} \geq 0$.

LEMMA : $\ln x + 1 > \frac{x}{x+1}$ if $x > 0$

PROOF OF LEMMA : Let

$$H(x) = \ln(x + 1) - \frac{x}{x + 1} \qquad (4-6)$$

Then $H(0) = 0$, and

$$H'(x) = \frac{1}{x + 1} - \frac{1}{(x + 1)^2} \qquad (4-7)$$

Whenever $x > 0$, then $\frac{1}{x+1} < 1$, so that the right side of equation $(4-7)$ is positive, meaning that $H(x) > 0$ for all $x > 0$, proving the lemma. •

So the lemma shows that $\ln(x + 1) > \frac{x}{x+1}$ if $x > 0$, which shows that $G'(1) > 0$, which means that $G(a) > 0$ whenever $a > 1$, which completes the proof of the theorem. •

Another way to show that the function has decreasing returns to scale is to use the formula for the elasticity of scale :

$$\mu(\mathbf{x}) = \mu_1(\mathbf{x}) + \mu_2(\mathbf{x}) = \frac{f_1 x_1}{f(\mathbf{x})} + \frac{f_2 x_2}{f(\mathbf{x})}$$

Here

$$\mu(\mathbf{x}) = \frac{(x_1/(x_1+1)) + (x_2/(x_2+1))}{\ln(x_1+1) + \ln(x_2+1)} \tag{4 – 8}$$

So $\mu(\mathbf{x}) < 1$ if and only if

$$\frac{x_1}{x_1+1} + \frac{x_2}{x_2+1} < \ln(x_1+1) + \ln(x_2+1) \tag{4 – 9}$$

Condition $(4-9)$ holds if

$$\frac{x}{1+x} < \ln(1+x) \tag{4 – 10}$$

for all $x > 0$.

And the Lemma above proves that $(4-10)$ must hold for all $x > 0$, so that $\mu(\mathbf{x}) < 1$ for all $\mathbf{x} >> 0$, so that the production function exhibits decreasing returns to scale.

Q5. Calculate the cost function $C(w_1, w_2, y)$ for the production function from question #4 above.

A5. Minimizing $w_1 x_1 + w_2 x_2$ subject to the constraint that $\ln(x_1+1) + \ln(x_2+1) > y$ means minimizing the Lagrangean

$$\mathcal{L} = w_1 x_1 + w_2 x_2 - \mu[\ln(x_1+1) + \ln(x_2+1) - y]$$

with first–order conditions

$$w_1 = \frac{\mu}{x_1+1} \tag{5 – 1}$$

$$w_2 = \frac{\mu}{x_2+1} \tag{5 – 2}$$

Equations $(5-1)$ and $(5-2)$ imply that

$$x_2 + 1 = \frac{w_1}{w_2}(x_1+1) \tag{5 – 3}$$

So that

$$y = \ln(x_1+1) + \ln(x_2+1) = \ln(x_1+1) + \ln[\frac{w_1}{w_2}(x_1+1)] = 2\ln(x_1+1) + \ln w_1 - \ln w_2 \tag{5 – 4}$$

or

$$\ln x_1 + 1 = \frac{1}{2}[y - \ln w_1 + \ln w_2] \tag{5 – 5}$$

Taking "anti–logarithms" of both sides of $(5-5)$

$$x_1 = \sqrt{\frac{e^y w_2}{w_1}} - 1 \tag{5 – 6}$$

which is the conditional input demand for input 1. Substituting from $(5-3)$,

$$x_2 = \sqrt{\frac{e^y w_2}{w_1}} - 1 \tag{5-7}$$

The cost of the cost–minimizing input combination is

$$C(w_1, w_2, y) = w_1 x_1 + w_2 x_2 = 2\sqrt{e^y w_1 w_2} - w_1 - w_2 \tag{5-8}$$

Differentiation of $(5-8)$ with respect to the input prices w_1 and w_2 yields the conditional input demand functions $(5-6)$ and $(5-7)$, so that Shephard's Lemma holds.