GS/ECON 5010 section "B"' Answers to Assignment 1 September 2011

$Q1$. Are the preferences described below transitive? Strictly monotonic? Convex? Explain briefly.

The person consumes 2 goods, food and clothing. A bundle A will be ranked as at least as good as bundle B if **either** of the following conditions holds :

(i) bundle A contains at least twice as much food as bundle B ;

(ii) bundle A contains at least half as much food as bundle B, and the amount of food in A added to the amount of clothing in A, is at least as large as the amount of food in B added to the amount of clothing in B.

If neither (i) or (ii) is true, then bundle A is not considered at least as good as bundle B.

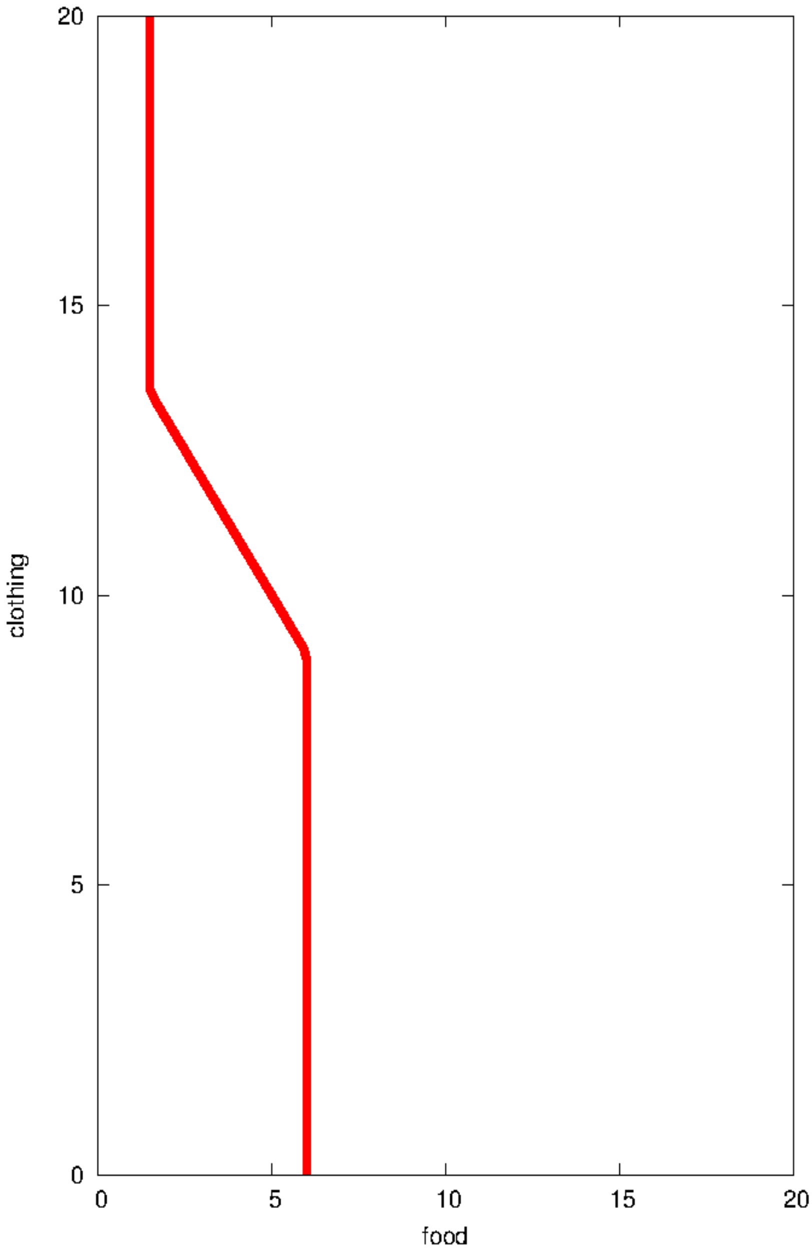
$A1$ First of all, the preferences are **not** transitive. To show this, what we need is a single example in which $A \succeq B$, and $B \succeq C$, but in which it is not true that $A \succeq C$.

Suppose that $A = (3, 8)$, $B = (5, 5)$ and $C = (7, 2)$, where the first number in each pair is the quantity of food in the bundle, and the second number is the quantity of clothing. Comparing A and B, part (ii) of the definition applies, so that A is at least as good as B. Comparing B and C, part (ii) of the definition applies again, so that B is at least as good as C. But the amount of food in C is more than twice as much as the amount of food in A. So A is **not** ranked as at least as good as C, which means the preferences are not transitive.

The preferences here **are** strictly monotonic. Suppose that bundle A contains strictly more food than bundle B, and strictly more clothing. Then the definition above says that A must be ranked as at least as good as B. But neither (i) nor (ii) holds when we compare B to A : B doesn't have twice as much food as A, and B does not have as much food plus clothing as A. So B is **not** at least as good as A. Saying A is at least as good as B, and B is not at least as good as A is exactly the same thing as saying A is strictly preferred to B. So if one bundle has strictly more of both goods, then it must be strictly preferred. (And if bundle A has at least as much of each good as B, then it must be at least as good as B, from part (ii) of the definition above.)

The preferences are **not** convex. Again, to show this, a single counter–example is sufficient. Suppose that $A = (3, 12)$, $B = (8, 0)$, and $C = (2, 16)$. Part (i) of the definition says that B is at least as good as A. Part (ii) of the definition says that C is at least as good as A. Now let D be a convex combination of B and C, the bundle that is halfway between B and C. Then $D = (5, 8)$. Bundle D does not contain at lest twice as much food as bundle A ; the amount of food in D, added to the amount of clothing in D, is less than the amount of food in A, added to the amount of clothing in A. So bundle D is **not** at least as good as A, even though it is a convex combination of bundles B and C. This example shows that preferences here are not convex, since $B \succeq A$, $C \succeq A$, but it is not true that $D \succeq A$, even though D is a convex combination of B and C.

Figure 1



(Figure 1 shows the set of bundles which are at least as good as $(3, 12)$: they are all the bundles on, or to the right of, the red line in the figure. This is not a convex set.)

*Q*2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1, x_2, x_3) = \max(x_1 + x_2 + x_3, 3x_1)$$

*A*2. These preferences are **not** convex. To show preferences are not convex (or not monotonic, or not transitive), it is sufficient to provide a single counter–example. So let $A = (3, 3, 3), B = (4, 0, 0), C = (0, 6, 6)$. The utility levels for the 3 bundles are $U(A) = 9, U(B) = 12, U(C) = 12$. So both B and C are at least as good as A, since they each yield higher utility. Now take a convex combination of B and C : $D \equiv 0.5B + 0.5C = (2, 3, 3)$. Since $U(D) = \max(8, 6) = 8$, therefore $U(D) < U(A)$. The bundle D is on a lower indifference surface than bundle A, even though D is a convex combination of two bundles (B and C) which are each on higher indifference surfaces than A.

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2) = \min(x_1 + x_2, 2x_2)$$

*A*3. Calculus is not that much help here, since the indifference curves for this utility function are kinked (as in the diagram) : if $x_1 > x_2$, then $u(x_1, x_2) = 2x_2$ and the slope of the indifference curve is 0 ; if $x_1 < x_2$, then $u(x_1, x_2) = x_1 + x_2$ and the slope of the indifference curve is -1. So, unless $p_1 = p_2$, the solution to the consumer's maximization is at a kink or a corner.

If $p_1 > p_2$, then the slope of the budget line is greater than the slope of the indifference curve, so that the consumer is at a corner : she maximizes her utility by spending all her money on good 2.

If $p_1 < p_2$, then the slope of the budget line is between the slopes of the two segments of the indifference curves. The consumer chooses the kink, at which $x_1 = x_2$.

If $p_1 = p_2$, then the consumer is indifferent among all points on the budget line for which $x_2 \geq x_1$: she does not have a unique Marshallian demand, because her preferences are not strictly convex.

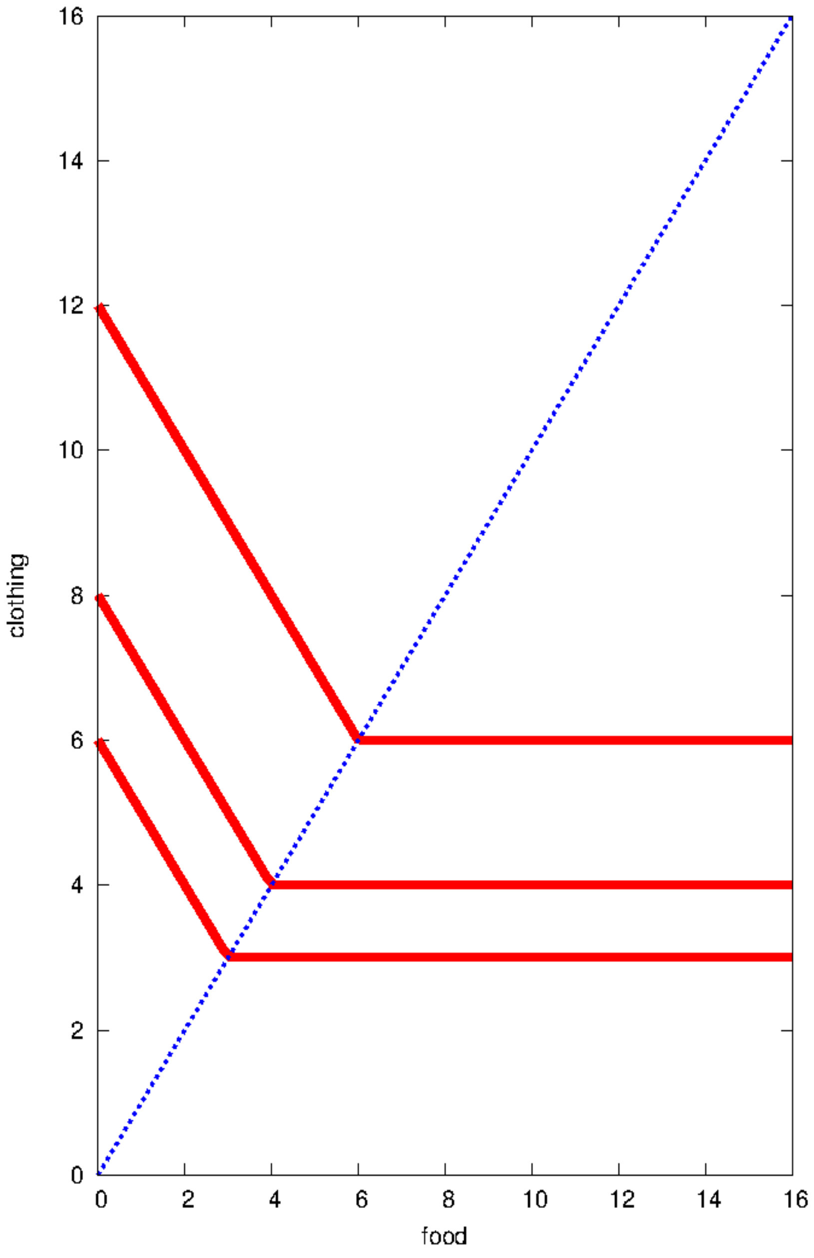
So her Marshallian demands are :

(*i*) if $p_1 > p_2$, then $x_1^M(p_1, p_2, y) = 0$ and $x_2^M(p_1, p_2, y) = y/p_2$.

(*ii*) if $p_1 = p_2$, then $x_1^M(p_1, p_2, y)$ is any x_1 less than or equal to $y/2p_1$, and $x_2^M(p_1, p_2, y) = \frac{y}{p_1} - x_1^M(p_1, p_2, y)$.

(*iii*) if $p_1 < p_2$, then $x_1^M(p_1, p_2, y) = \frac{y}{p_1+p_2} = x_2^M(p_1, p_2, y)$.

Figure 3



(Figure 3 : Indifference Curves for Question # 3)

*Q*4. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1 - \frac{1}{x_2} - \frac{4}{(x_3)^2}$$

*A*4. These are **quasi–linear** preferences, since the utility function is of the form $u(x_1, x_2, x_3) = ax_1 + f(x_2, x_3)$ (where here $a = 1$ and $f(x_2, x_3) = -\frac{1}{x_2} - \frac{4}{(x_3)^2}$). With quasi–linear preferences, the Marshallian demands for goods 2 and 3 will be independent of income.

To solve for the person's Marshallian demands, take the first–order conditions from the person's utility maximization, $u_i(x_1, x_2, x_3) = \lambda p_i$, for $i = 1, 2, 3$. Here, those conditions are

$$u_1(x_1, x_2, x_3) = 1 = \lambda p_1 \tag{4-1}$$

$$u_2(x_1, x_2, x_3) = \frac{1}{(x_2)^2} = \lambda p_2 \tag{4-2}$$

$$u_3(x_1, x_2, x_3) = \frac{8}{(x_3)^3} = \lambda p_3 \tag{4-3}$$

Equation $(4-1)$ implies that $\lambda = \frac{1}{p_1}$, so that equation $(4-2)$ can be written

$$\frac{1}{(x_2)^2} = \frac{p_2}{p_1} \tag{4-4}$$

or

$$x_2 = \sqrt{\frac{p_1}{p_2}} \tag{4-5}$$

Equation $(4-5)$ is the Marshallian demand function for good #2, since it expresses quantity demanded of good #2 as a function of prices and income. Substituting for λ into equation $(4-3)$ yields

$$\frac{8}{(x_3)^3} = \frac{p_3}{p_1} \tag{4-6}$$

or

$$x_3 = 2(\frac{p_1}{p_3})^{1/3} \tag{4-7}$$

which is the Marshallian demand function for good #3. To get the Marshallian demand function for good #1, the budget constraint must be used. Since $p_1x_1 + p_2x_2 + p_3x_3 = y$, therefore,

$$x_1 = \frac{y - p_2x_2 - p_3x_3}{p_1}$$

From equations $(4-5)$ and $(4-7)$

$$x_1^M(p_1, p_2, p_3, y) = \frac{y}{p_1} - \sqrt{\frac{p_2}{p_1}} - 2(\frac{p_3}{p_1})^{2/3} \tag{4-8}$$

Equations $(4-5)$, $(4-7)$ and $(4-8)$ define the Marshallian demands only if all three expressions have non–negative values. If p_1 is so high that expression $(4-8)$ is negative, then the person is at a corner solution. She chooses to consume none of good 1, and chooses x_2 and x_3 so as to maximize $-\frac{1}{x_1} - \frac{2}{(x_3)^2}$ subject to the constraint that $p_2x_2 + p_3x_3 \leq y$. The first–order conditions for this maximization imply that

$$p_2x_2 + 2(p_2p_3)^{2/3}x_2^{2/3} = y \tag{4 – 9}$$

which implicitly defines the Marshallian demand function for good 2, when good 1 is so expensive that the person chooses not to consume any of good 1. But I don't think that equation $(4-9)$ can be solved explicitly for x_2.

*Q*5. Calculate the Marshallian and Hicksian demand functions, the indirect utility function, and the expenditure function for a consumer whose preferences can be represented by the utility function

$$u(x_1, x_2) = \ln x_1 + \ln x_2 - \ln(x_1 + x_2)$$

*A*5. The first–order conditions from the consumer's utility maximization problem are

$$\frac{1}{x_1} - \frac{1}{x_1 + x_2} = \lambda p_1 \tag{5 – 1}$$

$$\frac{1}{x_2} - \frac{1}{x_1 + x_2} = \lambda p_2 \tag{5 – 2}$$

which can be re–arranged to

$$\frac{x_2}{x_1(x_1 + x_2)} = \lambda p_1 \tag{5 – 3}$$

$$\frac{x_1}{x_2(x_1 + x_2)} = \lambda p_2 \tag{5 – 4}$$

Setting the left side of $(5-3)$ divided by the left side of $(5-4)$ equal to the right side of $(5-3)$ divided by the right side of $(5-4)$,

$$(\frac{x_2}{x_1})^2 = \frac{p_1}{p_2} \tag{5 – 5}$$

or

$$x_2 = \sqrt{(\frac{p_1}{p_2})}x_1 \tag{5 – 6}$$

Substituting for x_2 from $(5-6)$ into the budget constraint $p_1x_1 + p_2x_2 = y$ yields

$$x_1(p_1 + \sqrt{(p_1p_2)}) = y \tag{5 – 7}$$

or

$$x_1 = \frac{y}{\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} \tag{5 – 8}$$

which is the Marshallian demand function for good #1. Substituting from $(5-8)$ into $(5-6)$ yields the Marshallian demand function for good #2,

$$x_2^M(p_1, p_2, y) = \frac{y}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \tag{5 − 9}$$

The first–order conditions for the "dual" problem, or minimizing the cost of a given utility level, are

$$\frac{\mu}{x_1} - \frac{\mu}{x_1 + x_2} = p_1 \tag{5 − 9}$$

$$\frac{\mu}{x_2} - \frac{\mu}{x_1 + x_2} = p_2 \tag{5 − 10}$$

These two equations again imply equation $(5-6)$ above. Substituting for x_2 from $(5-6)$ into the "given utility" constraint

$$\ln x_1 + \ln x_2 - \ln(x_1 + x_2) = \bar{u}$$

implies that

$$\ln x_1 + \ln\sqrt{p_1} - \ln\sqrt{p_2} + \ln x_1 - \ln(\sqrt{p_1} + \sqrt{p_2}) - \ln x_1 + \ln p_2 = \bar{u} \tag{5 − 11}$$

or

$$x_1 = e^{\bar{u}} \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_2}} \tag{5 − 12}$$

which is the Hicksian demand function for good 1. Substituting from $(5-12)$ into $(5-6)$ yields the Hicksian demand function for good #2,

$$x_2^H(p_1, p_2, \bar{u}) = e^{\bar{u}} \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}} \tag{5 − 13}$$

The expenditure function is defined as cost of the Hicksian demands,
$p_1 x_1^H(p_1, p_2, \bar{u}) + p_2 x_2^H(p_1, p_2, \bar{u})$. From equations $(5-12)$ and $(5-13)$, then

$$e(p_1, p_2, \bar{u}) = e^{\bar{u}}[\sqrt{p_1} + \sqrt{p_2}]^2 \tag{5 − 14}$$

The indirect utility can be calculated as

$$v(p_1, p_2, y) = \ln x_1^M(p_1, p_2, y) + \ln x_2^M(p_1, p_2, y) - \ln[x_M^1(p_1, p_2, y) + x_2^M(p_1, p_2, y)]$$

which (from equations $(5-7)$ and $(5-8)$) equals

$$v(p_1, p_2, y) = \ln y - 2\ln[\sqrt{p_1} + \sqrt{p_2}] \tag{5 − 15}$$

Equations $(5-14)$ and $(5-15)$ confirm that here $e(p_1, p_2, v(p_1, p_2, y)) = y$,
and $v(p_1, p_2, e(p_1, p_2, u)) = u$. As well, equations $(5-12)$, $(5-13)$ and $(5-14)$ show that Shepherd's lemma holds : $\partial e/\partial p_i = x_i^H(p_1, p_2, u)$ for $i = 1, 2$.

[These preferences are actually examples of *CES* preferences. Suppose that $\rho = -1$, in the textbook formulation of *CES* preferences. Then

$$U(x_1, x_2) = [x_1^{-1} + x_2^{-1}]^{-1}$$

The expression in the square brackets is $x_1^{-1} + x_2^{-1} = 1/x_1 + 1/x_2 = \frac{x_1+x_2}{x_1x_2}$. Taking this to the power -1 means inverting it, so that

$$U(x_1, x_2) = \frac{x_1x_2}{x_1 + x_2}$$

for *CES* preferences when $\rho = -1$. And the direct utility function $u(x_1, x_2) = \ln x_1 + \ln x_2 - \ln(x_1 + x_2)$ is the natural logarithm of this *CES* utility function $U(x_1, x_2)$.]