GS/ECON 5010 section "B"' Answers to Assignment 3 November 2011

$Q1$. What are the profit function, the supply function, and the (unconditional) input demand functions for a perfectly competitive firm with a production function

$$f(x_1, x_2) = \sqrt{\frac{x_1 x_2}{x_1 + 1}} \qquad ?$$

$A1$. If the production function has the equation $f(x_1, x_2) = \sqrt{\frac{x_1 x_2}{x_1+1}}$, then the partial derivatives are

$$f_1 = \frac{1}{2}(\frac{x_1 x_2}{1 + x_1})^{-1/2} \frac{x_2}{(x_1 + 1)^2} \qquad (1-1)$$

$$f_2 = \frac{1}{2}(\frac{x_1 x_2}{1 + x_1})^{-1/2} \frac{x_1}{x_1 + 1} \qquad (1-2)$$

Cost minimization requires that the ratio of marginal products equal the ratio of the input prices, so that $f_1/f_2 = w_1/w_2$, or

$$\frac{x_2}{x_1} \frac{1}{x_1 + 1} = \frac{w_1}{w_2} \qquad (1-3)$$

or

$$x_2 = \frac{w_1}{w_2} x_1(1 + x_1) \qquad (1-4)$$

Since $y = \sqrt{x_1 x_2/(1 + x_1)}$, equation $(1-4)$ implies that

$$y = \sqrt{\frac{w_1}{w_2}} x_1 \qquad (1-5)$$

if the firm minimizes costs. Equation $(1-5)$ can be re–arranged into the conditional input demand for x_1,

$$x_1 = \sqrt{\frac{w_2}{w_1}} y \qquad (1-6)$$

Substituting from $(1-4)$, the conditional demand for input 2 is

$$x_2 = y^2 + \sqrt{\frac{w_1}{w_2}} y \qquad (1-7)$$

so that the firm's cost function is

$$C(w_1, w_2, y) = w_2 y^2 + 2\sqrt{w_1 w_2} y \qquad (1-8)$$

Profit maximization by a competitive firm means maximization of $py - C(w_1, w_2, y)$ with respect to y, so that the firm's first–order condition for profit maximization is

$$p - 2w_2 y - 2\sqrt{w_1 w_2} = 0 \qquad (1-9)$$

or

$$y = \frac{p}{2w_2} - \sqrt{\frac{w_1}{w_2}} \qquad (1-10)$$

which is the firm's supply function.

The firm's profit function is $\pi(p, w_1, w_2) = py(p, w_1, w_2) - C[w_1, w_2, y(p, w_1, w_2)]$ which is (from equations $(1-8)$ and $(1-10)$)

$$\pi(p, w_1, w_2) = \frac{p^2}{4w_2} - p\sqrt{\frac{w_1}{w_2}} + w_1 \qquad (1-11)$$

From Hotelling's Lemma, the unconditional input demand functions are the negatives of the partial derivatives of the profit function :

$$x_1^u(p, w_1, w_2) = \frac{1}{2}\frac{p}{\sqrt{w_1 w_2}} - 1 \qquad (1-12)$$

$$x_2^u(p, w_1, w_2) = \frac{1}{4}[(\frac{p}{w_2})^2 - 2pw_1^{1/2}w_2^{-3/2}] \qquad (1-13)$$

*Q*2. What is the lowest price p for which a firm in perfect competition will be willing to produce a positive level of output, if the firm's cost function has the equation

$$C(w_1, w_2, y) = (\sqrt{w_1} + \sqrt{w_2})^2 \frac{y}{y+1} + w_1 y^2 \qquad ?$$

*A*2. The firm's supply curve is the upward sloping part of its marginal cost curve, where that curve is above the average cost curve. The firm's total cost function is

$$C(w_1, w_2, y) = (\sqrt{w_1} + \sqrt{w_2})^2 \frac{y}{y+1} + w_1 y^2$$

so that its marginal cost is

$$MC(w_1, w_2, y) = (\sqrt{w_1} + \sqrt{w_2})^2 \frac{1}{(y+1)^2} + 2w_1 y \qquad (2-1)$$

and the average cost function is

$$AC(w_1, w_2, y) = (\sqrt{w_1} + \sqrt{w_2})^2 \frac{1}{y+1} + w_1 y \qquad (2-2)$$

From $(2-1)$,

$$\frac{\partial MC}{\partial y} = -2(\sqrt{w_1} + \sqrt{w_2})^2 \frac{1}{(y+1)^3} + 2w_1 \qquad (2-3)$$

so that the marginal cost curve slopes down at $y = 0$ (and at small positive levels of output) whenever

$$(\sqrt{w_1} + \sqrt{w_2})^2 > 2w_1$$

which is equivalent to

$$w_2 > w_1 \tag{2-4}$$

Since $MC'' > 0$, the marginal cost curve slopes up for all values of $y \geq 0$ if $w_1 \geq w_2$, or is U–shaped, if $w_2 > w_1$.

The average cost curve is U–shaped :

$$\frac{\partial AC}{\partial y} = -(\sqrt{w_1} + \sqrt{w_2})^2 \frac{1}{(y+1)^2} + w_1 \tag{2-5}$$

which must be negative at $y = 0$.

The minimum average cost, where the average cost and marginal cost curves cross, occurs at an output level y where expression $(2-5)$ equals zero, or

$$y^* = \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_1}} - 1 \tag{2-6}$$

The value of AC=MC at an output level of y^* is (from equation $(2-1)$ or $(2-2)$)

$$MAC = w_1 + 2\sqrt{w_1 w_2} \tag{2-7}$$

The firm's supply curve is its marginal cost curve (defined by $(2-1)$, above its intersection with the average cost curve. At a price below the height of this intersection, the firm chooses not to produce. Therefore, the lowest price at which the firm is willing to supply a positive level of output is the minimum average cost defined by equation $(2-7)$.

*Q*3. Suppose that a market consists of an equal number of two types of people. Type–1 people's preferences can be represented by the utility function

$$U^1(x,z) = x + z - \frac{1}{2}z^2$$

and type–2 people's preferences can be represented by the utility function

$$U^2(x,z) = x + Vz - \frac{1}{2}z^2$$

where x and z are consumption of 2 different goods, and where $V > 1$. Good x is produced by a perfectly competitive industry, at a price of 1.

Good z is produced by a monopoly, which has **zero** production costs.

What price p should the monopoly charge so as to maximize its profits, if it must charge the same price p to all customers?

*A*3. If a person's utility function $U(x,z)$ is

$$U(x,z) = x + az - \frac{1}{2}z^2$$

then her Marshallian demand function for good z is

$$z = a - p \qquad \text{if} \quad p < a \tag{3-1}$$

if the price of good z is p, and the price of good x is 1, and her indirect utility function is

$$v(1, p, y) = y + \frac{1}{2}(a - p)^2 \tag{3-2}$$

(This indirect utility function will not be needed until question #4.)

If there N people each of types 1 and 2, then the monopoly's total sales will be

$$Z(p) = N(1 - p) + N(V - p) = N(1 + V - 2p) \tag{3-3}$$

since the demand functions of type 1 and type 2 people are $z_1 = 1 - p$ and $z_1 = V - p$ respectively.

So if the monopoly charges a price of p, then its total profits will be

$$N(1 + V - 2p)p \tag{3-4}$$

However, equation $(3-3)$ applies only if $p < 1$. If $p \geq 1$, then the type–1 consumers' demand will be driven down to zero. If the monopoly charges a price above 1, it will sell only to the high–demand type–2 consumers, its total sales will be $N(V - p)$ and its total profit will be

$$N(V - p)p \tag{3-5}$$

Note that the monopoly might choose to set such a high price ($p > 1$) that half its customers choose not to buy, if they are able to make enough money on sales to the remaining high–demand customers.

The derivative of profit with respect to the firm's price p is

$$\pi'(p) = N[1 + V - 4p] \qquad \text{if} \quad p < 1 \tag{3-6}$$

or

$$\pi'(p) = N[V - 2p] \qquad \text{if} \quad p > 1 \tag{3-7}$$

Note that this derivative jumps — from $V - 3$ to $V - 2$ — at the point $p = 1$. So there could be two local optima to the monnopoly's profit maximization problem.

The maximum of expression $(3-4)$ occurs at

$$p_1 = \frac{1 + V}{4} \tag{3-8}$$

and the maximum of expression $(3-5)$ occurs at

$$p_2 = \frac{V}{2} \tag{3-9}$$

So if $p_1 > 1$, the firm's best policy is to choose a high price, satisfying equation $(3-9)$. If $p_2 < 1$, the firm's best policy is to choose a low price, satisfying equation $(3-8)$. If $p_1 < 1 < p_2$, the firm must choose between the two local maxima.

From $(3-9)$, $p_2 \leq 1$ if

$$V < 2 \tag{3-10}$$

From $(3-8)$, $p_1 > 1$ if

$$V > 3 \tag{3-11}$$

If

$$2 < V < 3$$

the firm must choose between the two local maxima.

If it sets a price of p_1, then its profits will be

$$\pi_1 = N(1 + V - 2p_1)(p_1) = \frac{N}{8}(1 + V)^2 \tag{3-12}$$

whereas if it sets a price of p_2, its profits will be

$$\pi_2 = N(V - p_2)(p_2) = \frac{N}{4}V^2 \tag{3-13}$$

Therefore, $\pi_2 > \pi_1$ whenever

$$V > \frac{1}{\sqrt{2} - 1} \approx 2.4141 \tag{3-14}$$

If inequality $(3-14)$ holds, then the monopoly should choose a price p_2, and sell only to the high–demand (type 2) customers. Otherwise, it should choose a price p_1 and sell to both groups.

*Q*4. Suppose now that the monopoly in question #3 can charge a **two–part tariff**. That is, it requires all customers to pay a flat fee F in order to buy any of the good at all, along with a price p per unit.

[So customers have a choice : they either pay the fee F, and are able to buy as much or as little as they want of good z at a price of p per unit. Or they can decide not to pay the fee F, in which case they cannot purchase any of good z.]

The monopoly must charge the same fee F to all customers. And it must charge the same unit price p to all customers. If all other data are exactly as in question #3 above, what fee F and what unit price p should the monopoly charge?

*A*4. If there is a two–part tariff (F, p), consumers face a choice : either pay the fee and buy some of the monopolist's good, or refuse to pay the fee, and do without the monopolist's good. The consumer will choose to pay the fee if her utility is higher, that is if $v(1, p, y - F)$ is higher than the utility she would get if she spent all her money on the numéraire good x. This latter utility is simply her income y. So (from equation $(3-2)$), a type–1 consumer will choose to pay the fixed fee if and only if

$$y - F + \frac{1}{2}(1-p)^2 \geq y$$

which is equivalent to

$$F \leq \frac{1}{2}(1-p)^2 \qquad (4-1)$$

So suppose that the monopoly has chosen a price p, and a fee F, such that inequality $(4-1)$ holds. Then the higher–demand type–2 consumers must also be willing to pay the fee ; they are willing to pay the fee if and only if

$$F \leq \frac{1}{2}(V-p)^2 \qquad (4-2)$$

so that they are (strictly) more willing to pay the fee, since $V > 1$.

If inequality $(4-1)$ holds, then the monopoly's profit is

$$\Pi = 2NF + N(1+V-2p)p \qquad (4-3)$$

The first term on the right side of $(4-3)$ is the revenue from each customer's flat fee, and the second term is the profit from sales to those consumers.

Now suppose that the "participation constraint" $(4-1)$ held as a strict inequality ; that is, type–1 customers were strictly better off choosing to pay the fee and purchase the monopoly's good. Then the monopoly could raise its profit $(4-3)$ even further, simply by increasing the fee F a little, without changing p. Given the quasi–linear preferences of buyers, increasing the flat fee will not affect the monopoly's sales.

That means that the monopoly should never set the fee so low that $(4-1)$ holds as a strict inequality. If it wants type–1 customers to choose to participate, then it should set the fee exactly high enough to leave these customers on the margin of participation. So, given that it wants to sell to type–1 customers, it should choose some price $p \leq 1$, and a fee F equal to $(1-p)^2/2$, so that its profit is

$$\Pi_1 = N[(1-p)^2 + (1+V-2p)p] \qquad (4-4)$$

On the other hand, the monopoly may want to charge so high a fee that the type–1 consumers choose not to participate. In this case, only type–2 consumers will participate, so that the fee F must be low enough that their participation constraint $(4-2)$ holds, and the firm's profit will be

$$\Pi = NF + N(V-p)p \qquad (4-5)$$

Here too the firm will not want the participation constraint $(4-2)$ to hold as a strict inequality ; if type–2 consumers prefer strictly to participate, the monopoly can increase its profits by raising the fee F a little, holding constant the price. Therefore, it should set the fee F to satisfy $(4-2)$ with equality, and its profit, should it choose to sell only to type–2 consumers, will be

$$\Pi_2 = N[\frac{(V-p)^2}{2} + (V-p)p] \qquad (4-6)$$

So, as in question #3, the monopoly has two choices : set a low fee and serve both types of consumer, or set a high fee and serve only the high demand consumers. In the first case, it's picking p so as to maximize expression $(4-4)$, so that

$$p = \frac{V-1}{2} \tag{4-7}$$

meaning that its profits are (from equation $(4-4)$)

$$\Pi_1 = \frac{N}{4}[V^2 - 2V + 5] \tag{4-8}$$

However, if $V > 3$, then the price p defined by expression $(4-7)$ exceeds 1, which would be incompatible with type–1 consumers buying anything. (Why pay a positive fee for the right to buy the good at such a high price that you don't want to consume it?) If the monopoly wants to serve both types of consumer, and if $V > 3$, then it should set $p = 1$ and $F = 0$, so that its profits are

$$\Pi_1' = V - 1 \tag{4-9}$$

In the second case, it's picking p so as to maximize expression $(4-6)$, so that

$$p = 0 \tag{4-10}$$

and

$$\Pi_2 = \frac{V^2}{2} \tag{4-11}$$

In this latter case, it is giving away the product for free, so long as customers pay the fixed fee! Note first that low–demand customers will not be willing to pay the fee : even if the product is free, equation $(4-1)$ says they will never be willing to pay a fee which exceed $1/2$.

What's happening here (in this second case) is that the two–part tariff enables the monopoly to extract all the consumer surplus from the high–demand customers : it is charging them a fee equal to the area under their demand curve. And this result, that the price should equal the marginal cost, if only one type of consumer is being supplied, will be true as well if the marginal cost of production were positive. As long as the (identical) consumers' willingness to pay for a little more of the product exceeds the production cost, the monopoly can make more money by lowering its price, expanding its production, and charging a higher fee to the consumer to reflect the added benefit of consuming the good at a price which is less than the consumer's willingness to pay.

The monopoly will prefer this strategy – of charging a fee so high that only high–demand customers choose to buy – if and only if expression $(4-8)$ exceeds expression $(4-11)$, when $V < 3$. That's true if and only if

$$V > \sqrt{6} - 1 \approx 1.45 \tag{4-12}$$

The effects of being able to charge a flat admission fee F, in addition to a per–item charge p : (i) the monopoly's profits increase ; (ii) the monopoly is less likely to want to sell to the lower–taste customers ; (iii) the low–demand type–1 consumers are made worse off (or no better off).

The high–demand type–2 consumers may actually be better off in the presence of the fixed fee, but only if the monopoly chooses not to exclude the low–demand types. Here, if V is between 1.25 and 1.45, the high–demand types actually benefit from the lower unit price caused by the presence of the fixed fee, even taking into account the fee they pay.

*Q*5. In a Cournot oligopoly, in which n identical firms each have a constant marginal cost of production c, and in which market demand is defined by the inverse demand function

$$p = A - \frac{1}{\beta}(\sum_{i=1}^{n} x_i)^{\beta}$$

($\beta \neq 0$) for what values of β will it be true that the total industry output, in a symmetric Cournot equilibrium, increases with the number n of firms?

*A*5. Given the constant marginal cost, and given the inverse demand function, the profit $p(X)x_1 - cx_1$ of firm 1 can be writen

$$[A - \frac{1}{\beta}(\sum_{i=1}^{n} x_i)^{\beta}]x_1 - cx_1 \qquad (5-1)$$

Maximizing expression $(5-1)$ with respect to its own output x_1, firm #1 has a first–order condition

$$[A - \frac{1}{\beta}(\sum_{i=1}^{n} x_i)^{\beta} - c] - (\sum_{i=1}^{n} x_i)^{\beta-1}x_1 = 0 \qquad (5-2)$$

when we take into account the fact that firm #1's output is part of the industry aggregate output $\sum x_i$ which determines the price.

But for expression $(5-2)$ to represent firm #1's reaction function, the second–order conditions for a maximum must be satisfied. Differentiating the left side of expression $(5-2)$ with respect to x_1, the second–order condition is

$$2(\sum_{i=1}^{n} x_i)^{\beta-1} + (\beta - 1)(\sum_{i=1}^{n} x_i)^{\beta-2}x_1 \geq 0 \qquad (5-3)$$

Notice that this condition $(5-3)$ must be satisfied if $\beta > 1$, but otherwise it might not.

In a symmetric equilibrium, each firm i chooses that same output level x, and that output level is on the firm's reaction function. The first–order condition then becomes

$$[A - \frac{1}{\beta}X^{\beta} - c] - X^{\beta-1}\frac{X}{n} \qquad (5-4)$$

if $X \equiv nx$ is the total industry output.

Equation $(5-4)$ can be written

$$X^{\beta}[\frac{1}{n} + \frac{1}{\beta}] = A - c$$

or

$$X^{\beta}[\frac{n+\beta}{n\beta}] = A - c \qquad (5-5)$$

Equation $(5-5)$ defines the industry's total output (in a symmetric Cournot equilibrium) as a function of A, c, n and β.

Differentiating equation $(5-5)$ implicitly with respect to X and N shows the influence of the number of firms n on the aggregate output X :

$$X^{\beta-1}[\frac{n+\beta}{n}]\frac{\partial X}{\partial n} = \frac{X^{\beta}}{n^2} \qquad (5-6)$$

So the overall output X will increase with the number of Cournot oligopolists n if and only if

$$\beta + n > 0 \qquad (5-7)$$

Equation $(5-7)$ says that the industry output might actually **decrease** with the number of firms, if β were negative and if n were small.

However, we have to be sure that there is a symmetric Cournot equilibrium. If firms are maximizing their profits when X satisfies $(5-6)$, the second–order condition for maximization must be satisfied. At a symmetric Cournot equilibrium, condition $(5-3)$ becomes

$$2n + \beta - 1 \geq 0 \qquad (5-8)$$

If the number of firms is low enough (or β is negative enough) so that $(5-8)$ does not hold, then we do not have a symmetric Cournot equilibrium. (There may be an asymmetric Cournot equilibrium, or there may be no equilibrium at all.)

So is it possible that industry output X decreases with the number of firms n, in a symmetric Cournot equilibrium, in this model? This possibility arises if and only if condition $(5-8)$ holds (so that we have a symmetric equilibrium) but condition $(5-7)$ does not (so that output falls with the number of firms). But there are values of n and β for which this possibility occurs : if $\beta < 0$ and $n < -\beta < 2n-1$. For example, $\beta = -5$, and n between 3 and 5.