$Q1$. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of E_1 units of good 1 and E_2 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = 100 - \frac{1}{x_1^1} - \frac{1}{x_2^1}$$

$$u^2(x_1^2, x_2^2) = 50 - \frac{1}{x_1^2} - \frac{4}{x_2^2}$$

where x_j^i is person i's consumption of good j? [The superscripts in the definition of u^2 are the person's name, "2", not "squared".]

$A1$. An allocation $(\mathbf{x}^1, \mathbf{x}^2)$, with $\mathbf{x}^i >> 0$, will be efficient if and only if the two people's marginal rates of substitution are equal, or

$$\frac{u_1^1}{u_2^1} = \frac{u_1^2}{u_2^2} \qquad (1-1)$$

where u_j^i is person i's marginal utility from good j.

Here, that means that

$$\frac{(x_2^1)^2}{(x_1^1)^2} = \frac{(x_2^2)^2}{4(x_1^2)^2} \qquad (1-2)$$

or

$$\frac{x_2^1}{x_1^1} = \frac{x_2^2}{2x_1^2} \qquad (1-3)$$

Since $x_2^2 = E_2 - x_2^1$ and $x_1^2 = E_1 - x_1^1$, equation $(1-3)$ can be written

$$\frac{x_2^1}{x_1^1} = \frac{E_2 - x_2^1}{2(E_1 - x_1^1)} \qquad (1-4)$$

which defines x_1^2 as a function of x_1^1.

Equation $(1-4)$ can be written

$$x_2^1 = \frac{E_2 x_1^1}{2E_1 - x_1^1} \qquad (1-5)$$

which defines an upward–sloping curve in the Edgeworth box. Equation $(1-5)$ implies that $x_2^1 = 0$ when $x_1^1 = 0$, and that $x_2^1 = E_2$ when $x_1^1 = E_1$. So it goes through the corners of the Edgeworth box. That means that there are no Pareto optima to worry about in which consumption of some good by some person is zero.

The curve defined by $(1-5)$ lies everywhere below the diagonal in the Edgeworth box, since person #1 has a stronger taste for good 1 than good 2, compared with the other person #2.

*Q*2. What are the allocations in the core of the following 3–person, 2–good economy? Person i's preferences can be represented by the utility function $u^i(x_1^i, x_2^i)$, where

$$u^1(x_1^1, x_2^1) = x_1^1 x_2^1$$

$$u^2(x_1^2, x_2^2) = x_1^2 x_2^2$$

$$u^3(x_1^3, x_2^3) = x_1^3 + x_2^3$$

and the endowment vectors of the three people are $\mathbf{e}^1 = (3, 0), \mathbf{e}^2 = (1, 4), \mathbf{e}^3 = (2, 2)$.

*A*2. Note first that person #3's marginal rate of substitution between the two goods equals 1, independent of her consumption bundle, since she regards the 2 goods as perfect substitutes.

So in any Pareto optimal allocation of the 2 goods among the 3 people, it must be true that $MRS^1 = MRS^2 = MRS^3 = 1$. Now person 1 and person 2 have

$$MRS^i = \frac{x_2^i}{x_1^i} \quad i = 1, 2 \tag{2-1}$$

so that in any Pareto optimal allocation, it must be true that $x_1^1 = x_2^1$ and $x_1^2 = x_2^2$.

Since the overall endowment of the two goods is $E_1 = 6 = E_2$, for an allocation to be Pareto optimal, it must be of the form $\mathbf{x}^1 = (a, a)$, $\mathbf{x}^2 = (b, b)$, $\mathbf{x}^3 = (c, c)$, with $a + b + c = 6$. If an allocation is in the core, it must be Pareto optimal[1]. So the allocations in the core are of the form $\{(a, a), (b, b), (c, c)\}$, with $a + b + c = 6$. What is needed is to find what restrictions on a, b and c are imposed by the requirement that the allocation not be blocked by any coalition.

Now it must be true that $c \geq 2$, because of the "individual rationality" constraint. Person #3 would block any allocation in which $c < 2$, by forming a 1–person coalition of herself. She can get utility of 4 on her own, so that any core allocation must give her utility of at least 4.

But if $c > 2$, then the allocation could be blocked by a coalition of person #1 and person #2. Suppose that $c > 2$, so that $a + b < 4$. #1 and #2 could form a coalition in which person #1 got (a, a) and person #2 got $(4 - a, 4 - a)$, since their total endowments of the two goods are 4 of each. This new allocation $\{(a, a), (4 - a, 4 - a)\}$ is just as good as $\{(a, a), (b, b)\}$ for person #1, and strictly better for person #2, if originally $a + b < 4$. So any allocation $\{(a, a), (b, b), (c, c)\}$, in which $c > 2$ (and in which $a + b + c = 6$) will be blocked by a coalition of person #1 and person #2.

Therefore, in any core allocation, $c = 2$: any allocation in which $c < 2$ would be blocked by person #3, and any allocation in which $c > 2$ would be blocked by a coalition of person #1 and person #2.

Now what range of values for a lead to allocations which cannot be blocked? To answer that, consider what person #1 can do in a coalition with person #3. This coalition has a total

[1] otherwise it could be blocked by a coalition of all 3 people

endowment of $(5,2)$. To get person #3 to join the coalition, person #1 must offer person #3 a utility of at least 4. So if she forms a coalition with person #3, she'd choose $x_1^3 + x_2^3 = 4$. That gives her a utility of

$$(5 - x_1^3)(2 - x_2^3) = (5 - x_1^3)(2 - [4 - x_1^3]) = (5 - x_1^3)(x_1^3 - 2) \qquad (2-2)$$

Maximizing expression $(2-2)$ with respect to x_1^3 yields $x_1^3 = 3.5$. So, if she were to form a coalition with person #3, what person #1 would do is offer person #3 the consumption bundle $(3.5, 0.5)$, leaving herself with $(5 - 3.5, 2 - 0.5) = (1.5, 1.5)$.

That means that we must have $a \geq 1.5$ in any core allocation. If $a < 1.5$, then the allocation can be blocked by a coalition of people #'s 1 and 3, and the allocation $\{\mathbf{x}^1, \mathbf{x}^3\} = \{(1.5, 1.5), (3.5, 0.5)\}$.

Next, what range of values for b lead to allocations which cannot be blocked? To answer that, consider what person #2 can do in a coalition with person #3. This coalition has a total endowment of $(6,3)$. To get person #3 to join the coalition, person #2 must offer person #3 a utility of at least 4. So if he forms a coalition with person #3, he'd choose $x_1^3 + x_2^3 = 4$. That gives him a utility of

$$(6 - x_1^3)(3 - x_2^3) = (6 - x_1^3)(3 - [4 - x_1^3]) = (6 - x_1^3)(x_1^3 - 1) \qquad (2-3)$$

Maximizing expression $(2-3)$ with respect to x_1^3 again yields $x_1^3 = 3.5$. So, if he were to form a coalition with person #3, what person #2 would do is offer person #3 the consumption bundle $(3.5, 0.5)$, leaving himself with $(6 - 3.5, 3 - 0.5) = (2.5, 2.5)$.

That means that we must have $b \geq 2.5$ in any core allocation. If $b < 2.5$, then the allocation can be blocked by a coalition of people #'s 2 and 3, and the allocation $\{\mathbf{x}^2, \mathbf{x}^3\} = \{(2.5, 2.5), (3.5, 0.5)\}$.

So, so far, the restrictions on an allocation imposed by the possibility of blocking are : $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(a,a), (b,b), (c,c)\}$, with $c = 2$, $a \geq 1.5$, $b \geq 2.5$ and $a + b + c = 6$. There is exactly **one** such allocation, the allocation $\{(1.5, 1.5), (2.5.2.5), (2,2)\}$. So the core consists of exactly that one allocation. The blocking possibilities outlined above show that the core can't be any bigger than this one allocation. And the theorem that the competitive equilibrium is always inside the core shows that the core can't be any smaller than this one allocation.

*Q*3. What is the competitive equilibrium allocation for an exchange economy with a continuum of people, where the preferences of a type–v person can be represented by the utility function

$$u^v(x_1, x_2) = (x_1)^v (x_2)^{1-v}$$

where the taste type v is distributed uniformly over the interval $[0,1]$ (so that the fraction of people with a taste type of v or less is just v), and where each person has the same endowment of goods,

$$\mathbf{e} = (1, e_2) \qquad ?$$

*A*3. First, what is the excess demand function of a type–v person? This person has Cobb–Douglas preferences,

$$u(x_1, x_2) = x_1^v x_2^{1-v} \tag{3 – 1}$$

so that her Marshallian demand function for good #1 is

$$x^M(p_1, p_2, y) = \frac{vy}{p_1} \tag{3 – 2}$$

Here her income is the value of her endowment, which is $p_1 + p_2 e_2$, since her endowment vector is $(1, e_2)$. Substituting $y = p_1 + p_2 e_2$ into equation $(3 - 2)$, her demand function for good #1 is

$$x^M(p_1, p_2, p_1 + p_2 e_2) = v[1 + \frac{p_2}{p_1} e_2] \tag{3 – 3}$$

Adding up over all people in the economy, the total demand for good #1 is

$$X_1^D = \int_0^1 v[1 + \frac{p_2}{p_1} e_2] dv = [1 + \frac{p_2}{p_1} e_2] \int_0^1 v dv \tag{3 – 4}$$

Since v is the integral of $v^2/2$, therefore

$$X_1^D = \frac{1}{2}[1 + \frac{p_2}{p_1} e_2] \tag{3 – 5}$$

The aggregate endowment of good #1 is 1, since every person has the same endowment 1 (and the total population is 1). Therefore, the excess demand for good 1 is

$$Z_1 = X_1^D - 1 = \frac{1}{2}[\frac{p_2}{p_1} e_2 - 1] \tag{3 – 6}$$

The market for good #1 will be in equilibrium if $Z_1 = 0$, which implies then that

$$\frac{p_1}{p_2} = e_2 \tag{3 – 7}$$

Walras's Law says that if $Z_1 = 0$, then $Z_2 = 0$. Since only relative prices matter, the equilibrium prices for this exchange economy are any prices (p_1, p_2) for which $p_1/p_2 = e_2$.

What is the equilibrium allocation? A type–v person, facing prices (p_1, p_2) with $p_1/p_2 = e_2$, chooses to consume (from equation $(3 - 3)$)

$$x_1^v = 2v \tag{3 – 8}$$

Her consumption of the other good is

$$x_2^v = (1 - v)\frac{p_1 + p_2 e_2}{p_2} = 2(1 - v)e_2 \tag{3 – 9}$$

You can check that this allocation is feasible : $\int_0^1 x_1^v dv = 1$ and $\int_0^1 x_2^v dv = e_2$.

*Q*4. Give an example of a constant–sum (“zero–sum”) game which has exactly one Nash equilibrium in pure strategies.

*A*4. Suppose that we want the (unique) Nash equilibrium outcome in the top left corner of a 2–by–2 strategic form payoff matrix. For simplicity, let the payoffs in this top left corner be $(0, 0)$. For player #1 to pick the top row (when player #2 picks the left column), we need the payoff in the bottom left corner to be negative for player #1. Similarly, for player #2 to pick the left column when player #1 picks the top row, we need the payoff in the top right corner to be positive for player #1 (which means it’s negative for player #2).

Now just put the $(0, 0)$ payoff in the bottom right corner, and we’re done : the fact that the payoff for player #1 is positive in the top–right corner, and negative in the bottom–left corner, and the fact that the matrix is zero–sum, ensure that the bottom–right corner is not a Nash equilibrium.

So, for example

$$\begin{pmatrix} (0,0) & (1,-1) \\ (-1,1) & (0,0) \end{pmatrix}$$

will do.

*Q*5. Find all the Nash equilibria (in pure and mixed strategies) in the following strategic–form two–person game.

	a	b	c
A	$(4, 0)$	$(2, 2)$	$(2, 4)$
B	$(6, 4)$	$(12, 6)$	$(1, 8)$
C	$(5, 3)$	$(3, 12)$	$(0, 6)$
D	$(8, 6)$	$(6, 2)$	$(1, 2)$

*A*5. The strategy pairs (A, c) and (D, a) are both pure–strategy Nash equilibria : if player #1 plays A, then player #2 cannot increase his payoff by moving left from c to a or b, and if player #2 plays c, then player #1 cannot increase her payoff by moving down to B, C, or D. [Similarly, neither player wants to deviate unilaterally from (D, a).]

And these are the only pure–strategy Nash equilibria.

To find any mixed–strategy Nash equilibria might be complicated with a 4–by–3 payoff matrix. But the following result is useful :

result : if strategy s of a player can be “crossed off” during any repeated elimination of weakly dominated strategies, then strategy s will never be played with positive probability in any mixed–strategy Nash equilibrium

Note that row C is strictly dominated by row B for player #1. So C is a strictly dominated strategy, and will never be played with positive probability by player #1.

In the "reduced" payoff matrix obtained after crossing off row C, column b is weakly dominated by column c. So we can cross off column b if we're doing repeated elimination of weakly dominated strategies.

Once column b is gone, row B is now weakly dominated by row D, so we can cross it off in an iterated elimination of weakly dominant strategies.

Then the result above says that the only possible mixed–strategy Nash equilibria are those in which player #1 puts positive probability only on strategies A and D, and in which player #2 puts positive probability only on strategies a and c.

Could there be a mixed–strategy Nash equilibrium in which these 4 strategies are each played with strictly positive probability? When would player #1 be willing to mix between strategies A and D? Only if they each gave the same expected payoff to her. If player #2 were to play column a with probability β and column c with probability $1-\beta$, the expected payoff to player #1 from strategies A and D would be

$$\pi_A^1 = 4\beta + 2(1-\beta) \qquad (5-1)$$

$$\pi_D^1 = 8\beta + (1-\beta) \qquad (5-2)$$

so that player #1 will get the same expected payoff from both strategies A and D if and only if $\beta = 1/5$.

When would player #2 be willing to mix between columns a and c? Only if they each gave the same expected payoff to him. If player #1 were to play row A with probability α and row D with probability $1-\alpha$, the expected payoff to player #2 from strategies a and c would be

$$\pi_a^2 = 6(1-\alpha) \qquad (5-3)$$

$$\pi_c^2 = 4\alpha + 2(1-\alpha) \qquad (5-4)$$

so that player #2 will get the same expected payoff from both strategies a and c if and only if $\alpha = 1/2$.

Therefore there is exactly one Nash equilibrium in mixed strategies, in which player #1 uses the mixed strategy $(0.5, 0, 0, 0.5)$, and player #2 uses the mixed strategy $(0.2, 0, 0.8)$, where the vectors are the probability weights on strategies A, B, C and D for player #1, and on strategies a, b and c for player #2.