GS/ECON 5010 section "B"' Answers to Assignment 1 F2012

*Q*1. Are the preferences described below transitive? Continuous? Strictly monotonic? Explain briefly.

The person consumes 3 goods, white shirts (w), blue shirts (b), and green shirts (g). A bundle $A = (w, b, g)$ will be ranked as at least as good as bundle $A' = (w', b', g')$ if **any** of the following conditions holds :

(i) bundle A contains more shirts than bundle A' (i.e. $w + b + g > w' + b' + g'$) ; or

(ii) bundles A and A' contain the same number of shirts, but bundle A contains more white shirts ($w + b + g = w' + b' + g'$ and $w > w'$) ; or

(iii) bundles A and A' contain the same number of shirts **and** bundles A and A' contain the same number of white shirts **and** bundle A contains at least as many blue shirts ($w + b + g = w' + b' + g'$ and $w = w'$ and $b \geq b'$)

If neither (i) nor (ii) nor (iii) is true, then bundle A is not considered at least as good as bundle A'.

*A*1. These preferences **are** transitive : if bundle A has at least as many shirts as bundle A', and bundle A' has at least as many shirts as bundle A'', then bundle A must have at least as many shirts as bundle A''. Similarly, if bundle A has at least as many white shirts as bundle A', and bundle A' has at least as many white shirts as bundle A'', then bundle A must have at least as many white shirts as bundle A''. And if bundle A has at least as many blue shirts as bundle A', and bundle A' has at least as many blue shirts as bundle A'', then bundle A must have at least as many blue shirts as bundle A''.

The preferences **are** strictly monotonic : if $w \geq w'$ and $b \geq b'$ and $g \geq g'$, parts (i), (ii) and (iii) of the definition imply that $A = (w, b, g)$ must be ranked as at least as good as $A' = (w', b', g')$. If $w > w'$ and $b > b'$ and $g > g'$, then $w + b + g > w' + b' + g'$ so that part i of the definition implies that (w, b, g) be strictly preferred to (w', b', g').

But the preferences are **not** continuous. Take the bundle $A = (10, 0, 0)$. For **any** $\epsilon > 0$, the bundle $B(\epsilon) \equiv (5, 5 + \epsilon, 0)$ is strictly preferred to A (from part i of the definition), so that $B(\epsilon)$ is in the set $\succeq (A)$ of bundles which are at least as good as A. if this set $\succeq (A)$ is closed, then if we take the limit $B(0)$ of a sequence of bundles $B(\epsilon)$ in $\succeq (A)$, that limit must be in $\succeq (A)$: that's the definition of a closed set.

But that's not true here : the bundles $(5, 5.1, 0)$, $(5, 5.01, 0)$, $(5, 5.001, 0)$ and so on are all in $\succeq (A)$, from part i of the definition. But the bundle $B(0) = (5, 5, 0)$ is **not** in $\succeq (A)$, since $B(0)$ has the same number of shirts as A, but fewer white shirts.

[Another way of seeing that preferences are not continuous : What other bundles are on the indifference curve through some allocation, say $A = (3, 3, 4)$? If $B = (w, b, g)$ is on the same indifference curve as A, part (i) of the definition says that it must be true that $w + b + g = 10$; part (ii) says that it must be true that $w = 3$, and part (iii) says that $b = 3$. So the **only** consumption bundle on the indifference curve through $A = (3,3,4)$ is A itself. So that means that there must be points on the **boundary** of the "at least as good as" set $\succeq (A)$ which are not actually in $\succeq (A)$. Here these are points such as $(3,2,5)$ (or any (w,b,g) with $w+b+g=10$ and $g>4$). So $\succeq (A)$ is not closed, so that preferences are not continuous.]

*Q*2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1,x_2,x_3) = \sqrt{(x_1+x_2)^2+x_3}$$

*A*2. These preferences **are** strictly monotonic, since the partial derivatives of $u()$ are all positive :

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} = (x_1+x_2)[(x_1+x_2)^2+x_3]^{-0.5} > 0$$

$$\frac{\partial u}{\partial x_3} = 0.5[(x_1+x_2)^2+x_3]^{-0.5} > 0$$

But the preferences are **not** convex. Take the two bundles $\mathbf{x} = (2,2,0)$ and $\mathbf{z} = (0,0,16)$. Then

$$u(\mathbf{x}) = u(\mathbf{z}) = 4$$

Now take a convex combination of $\mathbf{x}$ and $\mathbf{z}$: $\mathbf{y} = (0.5)\mathbf{x} + (0.5)\mathbf{z} = (1,1,8)$ Then $u(\mathbf{y}) = \sqrt{12} \approx 3.464$ so that the person is indifferent between $\mathbf{x}$ and $\mathbf{z}$, but prefers either of those bundles (strictly) to a convex combination of the 2 bundles.

[Another way of seeing that the preferneces are not convex : Since preferences are strictly monotonic, every 2-dimensional indifference curve must exhibit a diminishing marginal rate of transformation, if preferences are convex. That is, a **necessary** condition for convexity, when preferences are strictly monotonic, is that u_i/u_j falls as x_i increases and x_j falls, holding constant every other x_k, and holding $u(\mathbf{x})$ constant.

So fix x_1, and look at an indifference curve between x_2 and x_3. These are combinations (x_2,x_3) such that $x_2^2+x_3$ are constant. So the indifference curve has the equation $x_3 = C - x_2^2$ where C is a constan. That curve has the **wrong** shape : take $C = 20$, and points such as $(0,20)$, $(1,19)$, $(2,16)$,$(3,11)$,$(4,4)$ are on the curve, (here the first number is x_2 and the second is x_3), so that the curve gets steeper as we move down and to the right.]

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1,x_2,x_3) = x_1 + x_2 + \ln(x_3)$$

*A*3. Notice first that this person views goods 1 and 2 as **perfect substitutes** for each other. She will only consume a positive amount of good 1 if $p_1 \leq p_2$.

[Proof : suppose that $p_1 > p_2$, and that $x_1 > 0$. Then reducing x_1 by a small amount (say 0.1), and increasing x_2 by that same amount will keep her utility level the same, but save her some money ($(0.1)(p_1 - p_2)$ in the example.) She can use this extra money to buy a little more of good 3, and raise her utility. So the original consumption bundle, with $x_1 > 0$, cannot be her most–preferred bundle in the budget set if $p_1 > p_2$.]

Similarly, she will only choose $x_2 > 0$ if $p_1 \geq p_2$.

So, suppose that $p_1 > p_2$. The person's problem now is to choose (x_2, x_3) so as to maximize $x_2 + \ln x_2$ subject to the budget constraint $p_2x_2 + p_3x_3 = y$, since she will not want to buy any of good 1.

The first–order conditions for this maximization are

$$1 = \lambda p_2 \qquad (3-1)$$

$$\frac{1}{x_3} = \lambda p_3 \qquad (3-2)$$

where λ is the Lagrange multiplier on the budget constraint $p_2x_2 + p_3x_3$. Substituting for λ from $(3-1)$ into $(3-2)$ yields the demand function for good 3,

$$x_3^M(\mathbf{p}, y) = \frac{p_2}{p_3} \qquad (3-3)$$

and substitution from $(3-3)$ into the budget constraint yields the demand function for good 2,

$$x_2^M(\mathbf{p}, y) = \frac{y}{p_2} - 1 \qquad (3-4)$$

[If $y < p_2 < p_1$, then she is at a corner solution, in which $x_1^M = x_2^M = 0$ and in which $x_3^M(\mathbf{p}, y) = \frac{y}{p_3}$.]

On the other hand, if $p_2 > p_1$, then she won't buy any of good 2, so that her demand functions are $x_3^M(\mathbf{p}, y) = \frac{p_1}{p_3}$ and $x_1^M = \frac{y}{p_1} - 1$ [provided that $p_1 < y$ so that she is not at a corner solution].

If it happened that $p_1 = p_2 = p$ then she would not care how she divided her spending between goods 1 and 2, so long as $x_3 = \frac{p}{p_3}$, and $x_1 + x_2 = \frac{y}{p} - 1$ [if $p < y$].

*Q*4. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function (where the expression "$exp(a)$" means e^a)

$$u(x_1, x_2) = 1 - \exp(-x_1) - \exp(-x_2)$$

*A*4. The key rule from calculus here is the derivative of the exponential function :

$$\frac{d}{dx}(e^a) = e^a$$

That means that the partial derivatives of the utility function with respect to x_1 and x_2 are

$$u_1 = \exp(-x_1)$$

$$u_2 = \exp(-x_2)$$

so that the marginal rate of substitution between the goods is

$$MRS = \frac{u_1}{u_2} = \exp(x_2 - x_1) \qquad (4-1)$$

(where I have used the fact that $e^{a-b} = \frac{e^a}{e^b}$).

The first–order condition for utility maximization by the consumer is $MRS = p_1/p_2$, which here (using $(4-1)$) can be written

$$\exp(x_2 - x_1) = \frac{p_1}{p_2} \qquad (4-2)$$

or

$$\exp(x_2) = \frac{p_1}{p_2}\exp(x_1) \qquad (4-3)$$

Taking natural logarithms of both sides of $(4-3)$, and using the facts that $\ln(e^a) = a$ and that $\ln ab = \ln a + \ln b$,

$$x_2 = \ln\frac{p_1}{p_2} + x_1 \qquad (4-4)$$

Substituting for x_2 from $(4-4)$ into the budget constraint $p_1x_1 + p_2x_2 = y$, yields the Marshallian demand function for good 1,

$$x_1^M(p_1, p_2, y) = \frac{y}{p_1 + p_2} + \frac{p_2}{p_1 + p_2}\ln\frac{p_2}{p_1} \qquad (4-5)$$

and subsituting from $(4-5)$ into $(4-4)$ gives the Marshallian demand function for good 2,

$$x_2^M(p_1, p_2, y) = \frac{y}{p_1 + p_2} + \frac{p_1}{p_1 + p_2}\ln\frac{p_1}{p_2} \qquad (4-6)$$

[Expressions $(4-5)$ and $(4-6)$ are valid only when they are both non–negative. If $p_1 > p_2$, and if income were so low that

$$y < p_2 \ln\frac{p_1}{p_2}$$

then we would have a corner solution in which $x_1 = 0$ and $x_2 = y/p_2$. Similarly, if $p_2 > p_1$ and income were so high that

$$y < p_1 \ln\frac{p_2}{p_1}$$

then we would have a corner solution in which $x_1 = y/p_1$ and $x_2 = 0$.]

5. Calculate the Hicksian demand functions, and the expenditure function, for a consumer whose preferences can be represented by the utility function from the previous question,

$$u(x_1, x_2) = 1 - \exp(-x_1) - \exp(-x_2)$$

*A*5. Expenditure minimization also has the first–order condition $MRS = p_1/p_2$, so that equation $(4-3)$ also applies to a consumer minimizing the cost of achieving a given level of utility.

Plugging $(4-3)$ into the definition of the consumer's utility,

$$u = 1 - \exp(-x_1) - \frac{p_2}{p_1}\exp(-x_1) \qquad (5-1)$$

so that

$$\frac{p_1+p_2}{p_1}\exp(-x_1) = 1-u \qquad (5-2)$$

or

$$\exp(x_1) = \frac{1}{1-u}\frac{p_1+p_2}{p_1} \qquad (5-3)$$

Taking natural logarithms of both sides of equation $(5-3)$ yields the Hicksian demand function for good 1 :

$$x_1^H(p_1, p_2, u) = \ln(p_1+p_2) - \ln p_1 - \ln(1-u) \qquad (5-4)$$

and substituting from $(5-3)$ into $(4-4)$ gives the Hicksian demand function for good 2,

$$x_2^H(p_1, p_2, u) = \ln(p_1+p_2) - \ln p_2 - \ln(1-u) \qquad (5-5)$$

The expenditure function, $E(p_1, p_2, u)$ is the cost of the Hicksian demands, $p_1 x_1^H(p_1, p_2, u) + p_2 x_2^H(p_1, p_2, u)$. From $(5-4)$ and $(5-5)$,

$$E(p_1, p_2, u) = (p_1+p_2)\ln(p_1+p_2) - p_1\ln p_1 - p_2\ln p_2 - (p_1+p_2)\ln(1-u) \qquad (5-6)$$

From equation $(5-6)$, we can also find the indirect utility function. Equation $(5-6)$ can be written

$$y = (p_1+p_2)\ln(p_1+p_2) - p_1\ln p_1 - p_2\ln p_2 - (p_1+p_2)\ln(1-v(p_1, p_2, y) \qquad (5-6)$$

so that

$$1 - v(p_1, p_2, y) = (p_1+p_2) = p_1^{-\frac{p_1}{p_1+p_2}} p_2^{-\frac{p_2}{p_1+p_2}} e^{-\frac{y}{p_1+p_2}} \qquad (5-7)$$

or

$$v(p_1, p_2, y) = 1 - (p_1+p_2)p_1^{-\frac{p_1}{p_1+p_2}} p_2^{-\frac{p_2}{p_1+p_2}} e^{-\frac{y}{p_1+p_2}} \qquad (5-8)$$

The indirect utility function can also be obtained from substitution of the Marshallian demand functions $(4-5)$ and $(4-6)$ into the expression for the direct utility function

$$v(p_1, p_2, y) = 1 - \exp[-(x_1^M(p_1, p_2, y)] - \exp[-(x_2^M(p_1, p_2, y)] \qquad (5-9)$$

to get

$$v(p_1, p_2, y) = 1 - [(\frac{p_2}{p_1})^{-\frac{p_2}{p_1+p_2}} + (\frac{p_1}{p_2})^{-\frac{p_1}{p_1+p_2}}]e^{-\frac{y}{p_1+p_2}} \qquad (5-10)$$

Expressions $(5-8)$ and $(5-10)$ are identical (as they must be), since it is always true that

$$(\frac{a}{b})^{-\frac{a}{a+b}} + (\frac{b}{a})^{-\frac{b}{a+b}} = (a+b)a^{-\frac{a}{a+b}}b^{-\frac{b}{a+b}} \qquad (5-11)$$

for any $a, b > 0$.