GS/ECON 5010 section “B”’ Answers to Assignment 4 November 2012

$Q1$. What are the allocations in the core of the following 3–person, 2–good economy?

Each of the three people regards the two goods as **perfect complements** : her preferences can be represented by the utility function $u(x_1^i, x_2^i) = \min(x_1^i, x_2^i)$.

The endowments of the three people are $\mathbf{e}^1 = (1, 0), \mathbf{e}^2 = (2, 0), \mathbf{e}^3 = (0, 3)$.

$A1$. Since the two goods are perfect complements, efficiency requires that each person get the same quantity of either good. So any allocation in the core must be of the form $\mathbf{x}^1 = (a, a), \mathbf{x}^2 = (b, b), \mathbf{x}^3 = (c, c)$, where $a + b + c = 3$.

“Individual rationality” (coalitions of size 1) has no impact here, since none of the three people has an endowment containing both goods : so going off on one’s own, and consuming one’s own endowment, yields each person a utility of 0. Nothing can be blocked by a coalition of size 1.

Similarly, the coalition $\{1, 2\}$ cannot block any allocation, since that coalition has no endowment of good 2.

The coalition $\{1, 3\}$ has a total endowment of $(1, 3)$. So if the proposed allocation has $a+c < 1$, then it can be blocked by a coalition of $\{1, 3\}$: that coalition could allocate (a, a) to person 1, and $(1 - a, 3 - a)$ to person 3, which would be better for person 3, and at least as good for person 1, as the original allocation, if (and only if) $1 - a > c$.

The coalition $\{2, 3\}$ has a total endowment of $(2, 3)$. So if the proposed allocation has $b+c < 2$, then it can be blocked by a coalition of $\{2, 3\}$: that coalition could allocate (b, b) to person 2, and $(2 - b, 3 - b)$ to person 3, which would be better for person 3, and at least as good for person 2, as the original allocation, if (and only if) $2 - b > c$.

And that’s all the possible blocking coalitions. So an allocation $(\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3)$ will be in the core if and only if $\mathbf{x}^1 = (a, a), \mathbf{x}^2 = (b, b), \mathbf{x}^3 = (c, c)$, with a, b, c all non–negative, and with $a + c \geq 1$, $b + c \geq 2$. So, for example, the allocations $((1, 1), (2, 2), (0, 0))$ and $((0, 0), (0, 0), (3, 3))$ are both in the core ; so are the allocations $((1, 1), (0, 0), (2, 2))$ and $((0, 0), (2, 2), (1, 1)$.

*Q*2. Show that the following allocation is **not** in the core, in the 20-person economy described below : $x^i = (9, 9)$ for i odd, and $x^i = (11, 11)$ for i even.

In the economy, each person's preferences can be represented by the utility function

$$u^i(x_1^i, x_2^i) = x_1^i x_2^i$$

The endowment vectors are $e^i = (20, 0)$ for i odd, and $e^i = (0, 20)$ for i even.

*A*2. In the proposed allocation, the odd–numbered people are getting less than the even–numbered people. So they would like to block the proposed allocation. However, they need the endowment of good #2 which only the even–numbered people have.

The best way for the odd–numbered people to block the original allocation is to form a coalition which leaves out only one of the even–numbered people. So consider a coalition with 19 people : all the odd–numbered people and all but one of the even–numbered people. To get any even–numbered people to join this blocking coalition, the coalition must offer these people at least as much utility as they got in the original proposed allocation. Here that's $(11)(11) = 121$.

One way of doing this is to offer them the exact same consumption bundle as they would get in the original proposed allocation : $(11, 11)$. [It's not the most efficient way of blocking, but it will turn out to be good enough here.]

Offering $(11, 11)$ to each of 9 even–numbered people will require 99 units of good 1 and 99 units of good 2. The total endowment of this 19–person coalition is $(200, 180)$. So that leaves 101 units of good 1, and 81 units of good 2, for each of the 10 odd–numbered people in the coalition. If this remaining endowment is divided equally among these 10 people, they'll each get the consumption bundle $(10.1, 8.1)$.

The utility each odd–numbered coalition member would get from this proposed coalition will be $(10.1)(8.1) = 81.61$. That's (just) higher than the utility they got in the original proposed allocation, which was $(9)(9) = 81$.

So the proposed allocation can be blocked by a coalition consisting of all 10 odd–numbered people, and 9 of the even–numbered people, with each odd–numbered coalition member getting $(10.1, 8.1)$ and each even–numbered coalition member getting $(11, 11)$.

*Q*3. What is the competitive (Walrasian) equilibrium in an exchange economy in which there are 1 million people of type 1, and 1 million people of type 2, in which each type–1 person has an endowment vector $\mathbf{e}^1 = (3, 1)$, each type–2 person has an endowment of $\mathbf{e}^2 = (2, 2)$ and each person, of either type, has preferences which can be represented by the utility function

$$u^i(x_1^i, x_2^i) = x_1^i[x_2^i]^2 \qquad ?$$

*A*3. Since each person has Cobb–Douglas preferences, her Marshallian demand function for good 2 can be written

$$x_2^i(\mathbf{p}, y) = \frac{2}{3}\frac{y^i}{p_2} \tag{3-1}$$

Suppose that we make good #1 the numéraire, so that the price vector is some $\mathbf{p} = (1, p)$, where p is the relative price of good 2, compared to the price of good 1. Then each person's income is $y^i = e_1^i + pe_2^i$, so that

$$y^1 = 3 + p \quad y^2 = 2 + 2p$$

and the overall demands of each person for good 2 will be

$$x_2^1 = \frac{2}{3p}(3 + p) \quad x_2^2 = \frac{2}{3p}(2 + 2p)$$

In equilibrium, the total quantity demanded of good 2 must equal the total endowment. The total demand (divided by 1 million) is $x_2^1 + x_2^2$, and the total endowment (divided by 1 million) is $1 + 2 = 3$. So the market for good #2 will be in equilibrium if and only if total quantity demanded equals total endowment, or

$$\frac{2}{3p}[3 + p + 2 + 2p] = 3 \tag{3-2}$$

Multiplying both sides of equation $(3-2)$ by $3p$, it can be written

$$9p = 10 + 6p \tag{3-3}$$

or

$$p = \frac{10}{3}$$

So the Walrasian equilibrium price vector is

$$\mathbf{p} = (3, 10) \tag{3-4}$$

— or any vector (p_1, p_2) which is proportional to $(3, 10)$.

With this price vector, the incomes of the two types of people are $y^1 = 3(3) + 10 = 19$, and $y^2 = 2(3) + 2(10) = 26$. The consumption vectors for the two people are then their Marshallian demands for the two goods, corresponding to incomes of 19 and 26 and the price vector $(3, 10)$. Since

$$x_1^i = \frac{1}{3p_1}y^i \tag{3-5}$$

with these Cobb–Douglas preferences, equations $(3-1)$ and $(3-5)$ imply that the consumption bundles for the two people in equilibrium, their Walrasian equilibrium allocations, are

$$x^1 = (\frac{19}{9}, \frac{19}{15}) \quad x^2 = (\frac{26}{9}, \frac{26}{15}) \tag{3-6}$$

*Q*4. Find all the Nash equilibria (in pure and mixed strategies) in the following strategic–form two–person game.

	a	b	c	d
A	$(0,1)$	$(6,2)$	$(0,0)$	$(10,1)$
B	$(2,3)$	$(4,5)$	$(1,4)$	$(8,10)$
C	$(1,6)$	$(0,4)$	$(0,8)$	$(6,8)$

*A*4. This game can be solved by iterated elimination of strictly dominated strategies — which means that it has a unique Nash equilibrium.

First note that strategy C of player #1 is strictly dominated by strategy B : each of the first numbers in the second row of the matrix is strictly bigger than the corresponding number in the third row.

So, if rational, player 1 will never play strategy C. Player 2 will realize this (if rationality of the players is common knowledge), and therefore believe that there is zero chance that player 1 will ever play strategy C. That reduces the strategic form to

	a	b	c	d
A	$(0,1)$	$(6,2)$	$(0,0)$	$(10,1)$
B	$(2,3)$	$(4,5)$	$(1,4)$	$(8,10)$

But if player 1 is going to play only A or B, then both strategies a and c are strictly dominated by strategy b for player 2. Player 2 knows that playing a (or c) would make sense only if there is some chance that player 1 would play C, and he knows that if player 1 is rational then she would never want to play C.

So we can cross out these two strategies for player 2, reducing the strategic form diagram to

	b	d
A	$(6,2)$	$(10,1)$
B	$(4,5)$	$(8,10)$

In this new game, player 1 has a strictly dominant strategy, A, since A is a better response for her than B to either b or d. [And she knows that 2 will play only b or d, since she knows that 2 knows that a or c would be rational only if 1 played C, and 1 knows that 2 knows that 1 is rational, and would never play C.]

Since 1 plays her dominant strategy (to this reduced game) A, 2 will play his best response to A, which is b.

Elimination of strictly dominated strategies results in (A, b) as an equilibrium, and since the eliminated strategies at each stage were strictly dominated, (A, b) is the only Nash equilibrium (in pure or mixed strategies) to this game.

*Q*5. Find the subgame perfect Nash equilibrium to the following 2–player game.

The game has several stages. The 2 players are the owners (player 1) and the hockey players (player 2). In stage 1, player 1 gets to propose **shares** (s_1, s_2) of the available revenue, which is \$1 billion initially. So s_1 is the share of the revenue which goes to player 1, and $s_2 \equiv 1 - s_1$ is the share which goes to player 2.

Player 2 moves next. Player 1 can "accept" the original proposal, in which case the game ends, with payoffs of s_1 times 1 billion dollars for player 1, and s_2 times 1 billion dollars for player 2. Or player 2 can "reject" the initial proposal, and counter–propose a different split (t_1, t_2) of the revenue. However, due to the delay caused by the bargaining, if player 2 rejects the initial proposal, the available revenue will have shrunk, from \$1 billion, to \$800 million.

If player 2 has rejected the initial proposal, and made a counter–proposal, then player 1 gets to move again. Player 1 can "accept" player 2's counter–proposal, in which case the game ends, with payoffs of t_1 times 800 million to player 1, and t_2 times 800 million to player 2. Or player 1 can "reject" the counter–proposal, and make a new (third) proposal (u_1, u_2) for a split of the revenue. But due to the delay caused by the extended bargaining, if player 1 rejects this counter–proposal, the available revenue will have shrunk, from \$800 million, to \$600 million.

If the first two proposals have been rejected, there is a final move to the game. Player 2 gets to choose whether to accept player 1's new proposal (u_1, u_2), or to reject it. If the proposal is accepted, the game ends, with payoffs of u_i times \$600 million to player i. But if this last proposal is rejected, the game still ends. If this last proposal is rejected, player 2 will still get \$200 million (from playing in the Kontinental Hockey League), but player 1 will get a zero payoff, because the season will be cancelled.

*A*5. To find the subgame perfect equilibrium to this game, it should be solved by backwards induction, from the "bottom up".

So start with the last decision node (which may never be reached) : player 2 deciding whether to accept player 1's last offer of (u_1, u_2). Player 2 will get a payoff of 200 (all payoffs measured in hundreds of millions of dollars) if he rejects the last offer. So his strategy, for this last sub–game, is to reject player 1's final offer, unless it gives player 2 a payoff of at least 200. Since player 2 gets a payoff of $(600)u_2$ if he accepts the final offer, then his best strategy, should this decision node be reached, is "accept an offer (u_1, u_2) if and only if $u_2 \geq \frac{1}{3}$". (I am assuming here that ties are broken in favour of acceptance.)

Now go to the previous node. If player 1 gets to make a final proposal (u_1, u_2), she should propose $u_2 = \frac{1}{3}$ (and $u_1 = \frac{2}{3}$). If u_2 were any lower, the offer would be rejected, and player 1 would get nothing. If u_2 were any higher, player 1 is giving up revenue unnecessarily, since any offer of $u_2 = \frac{1}{3}$ or more will be accepted for sure.

So if player 1 gets to make a final offer, she will propose $\mathbf{u} = (\frac{2}{3}, \frac{1}{3})$ and will get a payoff of 400. So that means she should accept any offer (from player 2) which gives her at least 400 : that is $t_1 \geq 0.5$ (since her payoff if she accepts a counter–offer from player 2 will be $800t_1$).

That means that player 1's equilibrium action, should she be given a counter–proposal from player #2, is : "accept any offer with $t_1 \geq 0.5$; reject any offer with $t_1 < 0.5$ and counter–propose $(u_1, u_2) = (\frac{2}{3}, \frac{1}{3})$".

Now consider player 2's action at his initial decision node. If he rejects the original offer, and gets to propose a counter–offer (t_1, t_2), then he should propose the minimum acceptable counter–offer, $(t_1, t_2) = (0.5, 0.5)$. That counter–proposal will be accepted (if player 1 plays her equilibrium strategy at the subsequent node), yielding a payoff to player 2 of $(0.5)(800) = 400$.

So player 2 expects to get a payoff of 400 if he rejects the original proposal, and counter–proposes his best counter–proposal (of $(0.5, 0.5)$). That means he should reject any offer (s_1, s_2) from player 1, unless that offer gives him a payoff of at least 400. That means rejecting the original offer unless $s_2 \geq 0.4$. His equilibrium action, at his first decision node, is therefore "accept any offer with $s_1 \leq 0.6$; reject any offer with $s_1 > 0.6$ and counter–propose $(t_1, t_2) = (0.5, 0.5)$".

Finally, we get to the top of the diagram. What should player 1 do in her original move? She anticipates that any offer will be rejected if $s_1 > 0.6$ — and that she would then wind up accepting a counter–offer which gives her $(0.5)(800) = 400$ as a payoff. She is better off making the minimal (from player 2's perspective) acceptable offer : $(s_1, s_2) = (0.6, 0.4)$.

So the subgame perfect equilibrium strategies are : "propose $(s_1, s_2) = (0.6, 0.4)$ initially ; if the original offer is reject, accept any counter–proposal with $t_1 \geq 0.5$; reject any counter–proposal with $t_1 < 0.5$ and 'counter–counter–propose' $(u_1, u_2) = (\frac{2}{3}, \frac{1}{3})$" for player 1, and "accept any original offer with $s_1 \leq 0.6$; reject any offer with $s_1 > 0.6$ and counter–propose $(t_1, t_2) = (0.5, 0.5)$; accept a final offer (u_1, u_2) if and only if $u_2 \geq \frac{1}{3}$ if the final stage is reached" for player 2.