GS/ECON 5010 section "B"' Answers to Assignment 1 F2013

*Q*1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

The person consumes only bread and cheese. Each kilo of bread contains 500 calories and 50 grams of protein. Each kilo of cheese contains 2000 calories and 100 grams of protein.

The person needs 2000 calories per day, and 50 grams of protein, in order to survive. So she is indifferent among all consumption bundles which do not provide enough calories or protein for her to survive.

If two bundles provide enough calories and protein for her to survive, then she prefers (strictly) the bundle with the highest value for $c+100p$ where c is the number of calories provided and p the number of grams of protein. (If two bundles have the same value for $c+100p$, and if both bundles have at least 2000 calories and 100 grams of protein, then she is indifferent between them.)

*A*1. Because the person is said to be indifferent among all bundles which do not provide enough protein and calories to survive, her preferences are **not** strictly monotonic.

For example, the bundle $A=(0.1,0.1)$ (where the first element is the number of kilos of bread, and the second element is the number of kilos of cheese) provides her with $(0.1)(500)+(0.1)(2000)=250$ calories per day, which is not enough to survive. The bundle $B=(0.2,0.2)$ provides her with $(0.2)(500)+(0.2)(2000)=500$ calories per day, which is still not enough to survive. So the person is indifferent between A and B, since neither bundle provides her with enough calories to survive[1]. So for this person, even though B has strictly more of each good than bundle A, she is indifferent between the bundles, violating the definition of strict monotonicity.

Her preferences are strictly monotonic among all bundles in the set S of bundles which give her adequate nutrition. That set can be defined as $\mathcal{S}=\{(x_1,x_2)|500x_1+2000x_2\geq 2000 \quad \text{and} \quad 50x_1+100x_2\geq 50\}$. But the second constraint (the "adequate protein" constraint) actually doesn't matter : if $500x_1+2000x_2\geq 2000$, and if $(x_1,x_2)\geq 0$, then it must be true that $50x_1+100x_2\geq 50$. So that means that the set of bundles of x_1 kilos of bread, and x_2 kilos of cheese which give adequate nutrition can be defined as the set $\mathcal{S}=\{(x_1,x_2)|500x_1+2000x_2\geq 2000 \quad \text{and} (x_1,x_2)\geq 0\}$ (which is the set of all bundles above the downward–sloping red line in the diagram).

If some bundle A does provide her with enough calories and protein to survive[2], then the set of other bundles which she likes just as much as A is a convex set. If $A=(a_1,a_2)$, with $500a_1+2000a_2\geq 2000$ and $50a_1+100a_2\geq 50$, then some other bundle $B=(b_1,b_2)$ is at least as good as A if

$$500b_1+2000b_2+100(50b_1+100b_2)\geq 500a_1+2000a_2+100(50a_1+100a_2) \qquad (1-1)$$

[1] You can check that they don't provide her with enough protein either. But that's not really necessary, since she needs both protein and calories to survive.

[2] That is, if the bundle A is in the set $\mathcal{S}$ defined in the previous paragraph.

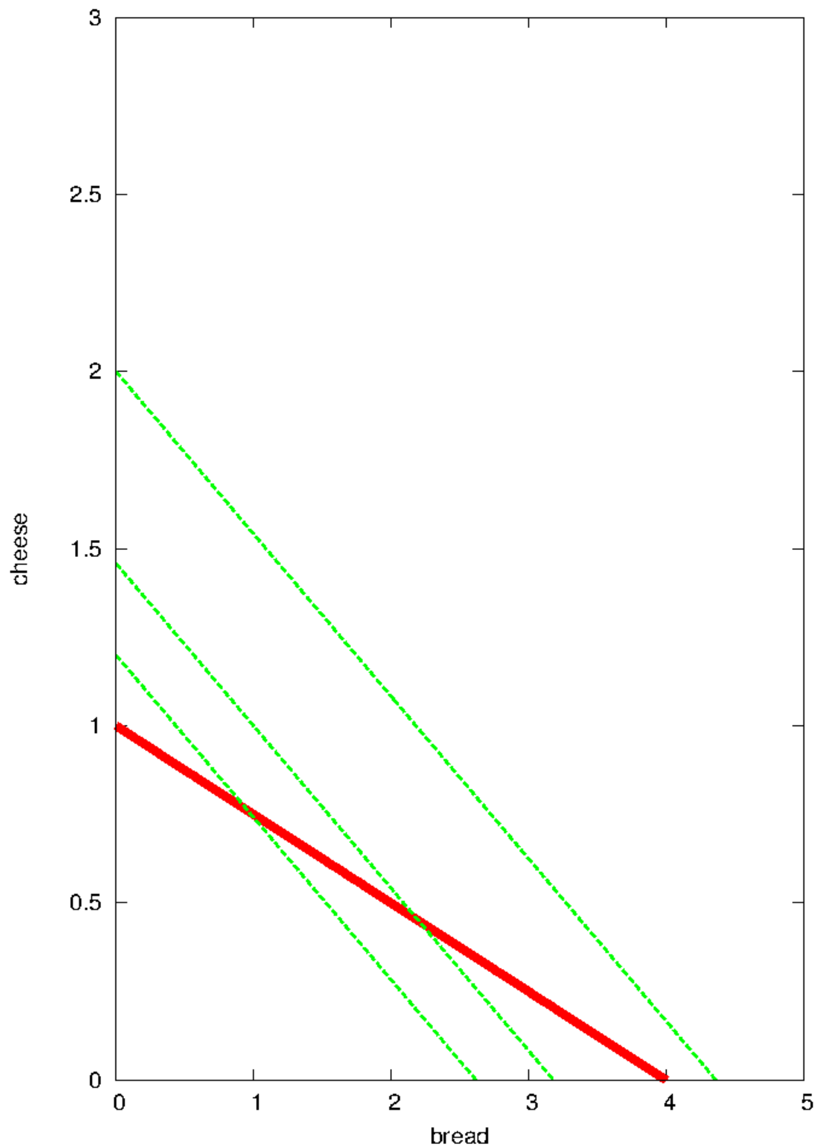

Figure 1 : In the figure, combinations of bread and cheese which are on or above the thick (red) line satisfy the minimum–calorie constraint.

The indifference curves — for bundles which satisfy the minimum calorie constraint — are bundles on the dashed (green) lines, where they are above the minimum calorie constraint (the thick (red) line).

So the bread–cheese combinations which are preferred strictly to the bundle $(1, 1)$ are all the bundles above the middle dashed (green) line through $(1, 1)$, which are also above the solid (red) line.

Inequality $(1-1)$ can be written

$$5500b_1 + 12000b_2 \geq 5500a_1 + 12000a_2 \qquad (1-2)$$

The set of (b_1, b_2) which satisfy $(1-2)$ are all those bundles which lie on or above a line, with slope $-5500/12000 = -11/24$. That's a convex set. So preferences look convex.

But wait. B is at least as good as A only if it satisfies two criteria : (1) it must give at least as high a value for $c + 100p$ as A, and (2) it must meet the basic needs for survival. Inequality $(1-2)$ defines only criterion #1. Criterion #2 is satisfied by the bundle B only if B is in the set $\mathcal{S}$ defined two paragraphs above.

So, if A is in $\mathcal{S}$, then B is at least as good as A if and only if it satisfies both criteria, that is if and only if it satisfies the two inequalities $(1-2)$ and $(1-3)$:

$$500b_1 + 2000b_2 \geq 2000 \qquad (1-3)$$

That means that preferences **are** convex, even taking into account the second criterion. When $A \in \mathcal{S}$, the set of bundles which are at least as good as A are the bundles which lie above both lines defined by inequalities $(1-2)$ and $(1-3)$. That's a convex set[3].

What if some bundle A does not provide the minimum level of protein or calories? Then the set of all bundles which are at least as good as A is the set of all bundles in $\mathcal{R}^2$: that's certainly a convex set.

So preferences are convex here, although they are not strictly monotonic.

*Q*2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1, x_2) = 20x_1 + 5x_2 - \frac{10}{x_2}$$

*A*2. The function $U(x_1, x_2)$ can be differentiated : its partial derivatives are

$$U_1(x_1, x_2) = 20 \qquad (2-1)$$

$$U_2(x_1, x_2) = 5 + \frac{10}{(x_2)^2} \qquad (2-2)$$

Since both partial derivatives are strictly positive, the preferences represented by the function $U(\cdot, \cdot)$ must be strictly monotonic.

[3] Proof : the set of bundles (x_1, x_2) which lie above some line $\alpha x_1 + \beta x_2 \geq \gamma$ (for some numbers α, β and γ) is called a "half–plane". Any half–plane is a convex set. And any intersection of several convex sets is also a convex set. So S is a convex set, since it's a half–plane. And the set of bundles satisfying $(1-2)$ and $(1-3)$ is convex, since it's the intersection of the convex set $\mathcal{S}$ and the convex half–plane defined by $(1-2)$.

Since the preferences are strictly monotonic, and since there are only two goods, preferences will be convex if and only if the indifference curves have the "usual" shape, getting less steep as we move down and to the right. If we graph consumption of good #1 on the horizontal axis, and consumption of good #2 on the vertical axis, then the slope of an indifference curve is

$$\frac{dx_2}{dx_1}_{U=\bar{U}} = -\frac{U_1}{U_2} = -\frac{20}{5+\frac{10}{(x_2)^2}} \tag{2-3}$$

As we move down and to the right, x_1 increases and x_2 decreases, so expression $(2-3)$ shows that the slope of the indifference curve decreases in absolute value.

Alternatively, equation $(2-3)$ can be written

$$\frac{dx_2}{dx_1}_{U=\bar{U}} = -\frac{20z}{5z+10} \tag{2-4}$$

where $z \equiv (x_2)^2$. The derivative of $(2-4)$ with respect to z is

$$-\frac{1}{(5z+10)^2}[20(5z+10) - 5(20z)] = \frac{-200}{(5z+10)^2} < 0 \tag{2-5}$$

So as we move down the indifference curve, x_2 decreases, so z decreases, so that the indifference curve gets less steep.

Therefore, since the indifference curves have the "usual" shape, preferences are convex.

[Alternatively, if we take the matrix of second derivatives of the utility function, this matrix has all zeros, except for a negative number in the bottom right corner. That means that it's a negative semi–definite matrix, so that the function $U(x_1, x_2)$ is a concave function, which means that preferences are convex.]

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = 12 - \frac{1}{x_1} - \frac{1}{\sqrt{x_2 x_3}}$$

*A*3. The first derivatives of the utility function are

$$U_1 = \frac{1}{(x_1)^2} \tag{3-1}$$

$$U_2 = \frac{1}{2}(x_2)^{-3/2}(x_3)^{-1/2} \tag{3-2}$$

$$U_2 = \frac{1}{2}(x_2)^{-1/2}(x_3)^{-3/2} \tag{3-3}$$

Since the conditions for utility maximization are $U_i = \lambda p_i$ $(i = 1, 2, 3)$ where λ is the Lagrange multiplier on the budget constraint, we get

$$\frac{1}{(x_1)^2} = \lambda p_1 \tag{3-4}$$

$$\frac{1}{2}(x_2)^{-3/2}(x_3)^{-1/2} = \lambda p_2 \qquad (3-5)$$

$$\frac{1}{2}(x_2)^{-1/2}(x_3)^{-3/2} = \lambda p_3 \qquad (3-6)$$

Dividing the left side of equation $(3-5)$ by the left side of equation $(3-6)$:

$$\frac{x_3}{x_2} = \frac{p_2}{p_3} \qquad (3-7)$$

or

$$x_3 = \frac{p_2}{p_3}x_2 \qquad (3-8)$$

Now equation $(3-5)$ can be re–written

$$\frac{1}{2}(x_2)^{-3/2}[\frac{p_2}{p_3}x_2]^{-1/2} = \lambda p_2 \qquad (3-9)$$

or

$$(x_2)^{-2} = 2\lambda(p_2)^{3/2}(p_3)^{1/2} \qquad (3-10)$$

Since equation $(3-4)$ implies that

$$\lambda = \frac{1}{p_1(x_1)^2} \qquad (3-11)$$

equation $(3-10)$ can be written

$$(x_2)^{-2} = 2(p_1)^{-1}(p_2)^{3/2}(p_3)^{1/2}(x_1)^{-2} \qquad (3-12)$$

which means that

$$x_2 = \frac{1}{\sqrt{2}}[(p_1)^{1/2)}(p_2)^{-3/4}(p_3)^{1/4}]x_1 \qquad (3-13)$$

and (from equation $(3-8)$)

$$x_3 = \frac{1}{\sqrt{2}}[(p_1)^{1/2)}(p_2)^{1/4}(p_3)^{-3/4}]x_1 \qquad (3-14)$$

Using the $(3-13)$ and $(3-14)$, the budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = y$ becomes

$$(p_1 + \frac{1}{\sqrt{2}}[(p_1)^{1/2)}(p_2)^{1/4}(p_3)^{1/4}] + \frac{1}{\sqrt{2}}[(p_1)^{1/2)}(p_2)^{1/4}(p_3)^{1/4}])x_1 = y \qquad (3-15)$$

which expresses the quantity demanded for good 1 as a function of the person's income, and the prices of the three goods. Re–arranging,

$$x_1^M(p_1, p_2, p_3, y) = \frac{y}{p_1 + \sqrt{2}[(p_1)^{1/2}p_2^{1/4}p_3^{1/4}]} \qquad (3-16)$$

and substitution from equations $(3-13)$ and $(3-14)$ yields

$$x_2^M(p_1, p_2, p_3, y) = \frac{y}{\sqrt{2}(p_1)^{1/2}(p_2)^{3/4}(p_3)^{-1/4} + 2p_2} \qquad (3-17)$$

and

$$x_3^M(p_1,p_2,p_3,y) = \frac{y}{\sqrt{2}(p_1)^{1/2}(p_2)^{-1/4}(p_3)^{3/4} + 2p_3} \qquad (3-18)$$

*Q*4. For what values of income y and prices (p_1, p_2, p_3) will a person demand strictly positive quantities of good #1, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1 + \sqrt{x_2 x_3} \qquad ?$$

*A*4. The key here is that the person regards good #1 as a **perfect substitute** for the aggregate $\sqrt{x_2x_3}$ of goods #2 and #3.

Now if we had just two goods, and they were perfect substitutes, then the person would choose to buy positive quantities of a good only if it were at least as cheap as the other good.

Here, good #1 is a perfect substitute for some combination of goods #2 and #3, so that she will want to buy good #1 only if it is cheaper than the best combination of the other two goods.

In particular, the condition that she be willing to buy any of good #1 is that

$$p_1 \leq 2\sqrt{p_2 p_3} \qquad (4-1)$$

To see this, take the partial derivatives of the utility function :

$$U_1 = 1 \qquad (4-2)$$

$$U_2 = \frac{1}{2}(x_2)^{-1/2}x_3^{1/2} \qquad (4-3)$$

$$U_3 = \frac{1}{2}(x_2)^{1/2}x_3^{-1/2} \qquad (4-4)$$

Now if we just use the usual "derivative" conditions for the consumer's utility maximization, we get

$$1 = \lambda p_1 \qquad (4-5)$$

$$\frac{1}{2}(x_2)^{-1/2}x_3^{1/2} = \lambda p_2 \qquad (4-6)$$

$$\frac{1}{2}(x_2)^{1/2}x_3^{-1/2} = \lambda p_3 \qquad (4-7)$$

Equations $(4-6)$ and $(4-7)$ imply that, if the consumer chooses positive quantities of goods #2 and #3, then

$$\frac{x_2}{x_3} = \frac{p_3}{p_2} \qquad (4-8)$$

or

$$p_2 x_2 = p_3 x_3 \qquad (4-9)$$

Equation $(4-9)$ says that any money spent on goods #2 and #3 should be split evenly between them. Not a big surprise, perhaps, since the aggregate $\sqrt{x_2x_3}$ of goods #2 and #3 looks like a

Cobb–Douglas utility function, and a person's share of expenditure on a good is fixed if she has Cobb–Douglas preferences.

So if the person chooses to buy x_1 units of good 1, which will cost her p_1x_1 dollars, then she will have $y - p_1x_1$ left to spend on the other two goods. Equation $(4-9)$ says she'll split that expenditure evenly between those goods, so that

$$x_2 = \frac{1}{2p_2}(y - p_1x_1) \tag{4 – 10}$$

$$x_3 = \frac{1}{2p_3}(y - p_1x_1) \tag{4 – 11}$$

Her overall level of utility — given that she bought x_1 units of good 1, and divided the rest of her money efficiently between the other two goods — will be

$$U = x_1 + \sqrt{[\frac{1}{2p_2}(y - p_1x_1)][\frac{1}{2p_3}(y - p_1x_1)]} \tag{4 – 12}$$

or

$$U = x_1 + \frac{y - p_1x_1}{2\sqrt{p_2p_3}} \tag{4 – 13}$$

Now what happens to her overall utility if she did the following? : (1) bought a little more of good 1 and (2) adjusted her consumption of goods #2 and #3 so as to keep condition $(4-9)$ satisfied? From $(4-13)$ the change in her overall utility would be

$$1 - \frac{p_1}{2\sqrt{p_2p_3}} \tag{4 – 14}$$

times the increase in x_1.

So if expression $(4-14)$ is positive, increasing her consumption of x_1 must increase her utility[4]. And if the expression is negative, any decrease in x_1, with the money saved split evenly between the other two goods, will increase her utility.

So if expression $(4-14)$ is negative, she should keep decreasing her consumption of good 1, splitting the money saved between the other two goods, until x_1 hits zero. Only if the expression is non–negative should she spend any money on good #1.

[An alternative way of seeing the result : if we substitute from equation $(4-8)$ for x_3, and from equation $(4-5)$ for λ, into equation $(4-6)$, we get

$$p_1 = 2\sqrt{p_2p_3} \tag{4 – 15}$$

But that equation can't hold — except in the very special case that $(4-14)$ happens to equal 0. What $(4-15)$ is telling us is that we should increase x_2, and decrease x_1, whenever the left side of $(4-15)$ exceeds the right – and keep doing that until we are at a corner solution at which $x_1 = 0$.

[4] if she continues to allocate her remaining income efficiently between the other 2 goods

Q5. Find the expenditure function and the Hicksian demand functions for a person whose direct utility function is

$$u(x_1, x_2) = \frac{x_1 x_2}{1 + x_2}$$

A4. The partial derivatives of this utility function are

$$U_1(x_1, x_2) = \frac{x_2}{1 + x_2} \qquad (5-1)$$

$$U_2(x_1, x_2) = \frac{x_1}{(1 + x_2)^2} \qquad (5-2)$$

If the consumer's indifference curve is tangent to a budget line, then it must be the case that $U_1/U_2 = p_1/p_2$, or

$$\frac{x_2(1 + x_2)}{x_1} = \frac{p_1}{p_2} \qquad (5-3)$$

which can be written

$$\frac{x_1}{1 + x_2} = \frac{p_2}{p_1} x_2 \qquad (5-3)$$

If $(5-3)$ is used to substitute for $x_1/(1 + x_2)$ in the utility function, then

$$u = \frac{p_2}{p_1}(x_2)^2 \qquad (5-4)$$

which expresses demand for good 2 as a function of prices, and the consumer's level of utility. Therefore, the Hicksian demand for good #2 is

$$x_2^H(p_1, p_2, u) = \sqrt{\frac{p_1 u}{p_2}} \qquad (5-5)$$

which implies that

$$1 + x_2 = \frac{\sqrt{p_2} + \sqrt{p_1 u}}{\sqrt{p_2}} \qquad (5-6)$$

so that

$$\frac{x_2}{1 + x_2} = \frac{\sqrt{p_1 u}}{\sqrt{p_1 u} + \sqrt{p_2}} \qquad (5-7)$$

Equation $(5-7)$ and the fact that $u = x_1 \frac{x_2}{1+x_2}$ imply that the Hicksian demand function for good #1 is

$$x_1^H(p_1, p_2, u) = u + \frac{\sqrt{p_2 u}}{\sqrt{p_1}} \qquad (5-8)$$

and the fact that $e(p_1, p_2, u) = p_1 x_1^H(p_1, p_2, u) + p_2 x_2^H(p_1, p_2, u)$ means that the expenditure function is

$$e(p_1, p_2, u) = p_1 u + 2\sqrt{p_1 p_2 u} \qquad (5-9)$$