GS/ECON 5010 section "B"' Answers to Assignment 2 October 2013

$Q1$. Could the function

$$v(\mathbf{p}, y) = \frac{1}{p_1 + p_2}[y + \frac{(p_1)^2 + (p_2)^2}{p_3} - 6p_3]$$

be an indirect utility function for some consumer with well–behaved preferences? Explain.

(You can assume that the person's income y is large enough, relative to prices, that the consumer's quantities demanded are non–negative.)

$A1$. If this function were an indirect utility function, the fact that $v(\mathbf{p}, e(\mathbf{p}, u)) = u$ implies that the expenditure function for these preferences would have to satisfy

$$\frac{1}{p_1 + p_2}[e(\mathbf{p}, u) + \frac{(p_1)^2 + (p_2)^2}{p_3} - 6p_3] = u \qquad (1-1)$$

or

$$e(\mathbf{p}, u) = u(p_1 + p_2) - \frac{(p_1)^2 + (p_2)^2}{p_3} + 6p_3 \qquad (1-2)$$

Notice that the "candidate" expenditure function $(1-2)$ is homogeneous of degree 1 in all prices : doubling all prices (p_1, p_2, p_3) will double the value of the function.

Differentiating $(1-2)$ with respect to prices yields

$$e_1(\mathbf{p}, u) = u - 2\frac{p_1}{p_3} \qquad (1-3)$$

$$e_2(\mathbf{p}, u) = u - 2\frac{p_2}{p_3} \qquad (1-4)$$

$$e_3(\mathbf{p}, u) = \frac{(p_1)^2 + (p_2)^2}{(p_3)^2} + 6 \qquad (1-5)$$

If $e(\mathbf{p}, u)$ is an expenditure function, Shepherd's Lemma says that these three derivatives, $e_1(\mathbf{p}, u)$, $e_2(\mathbf{p}, u)$ and $e_3(\mathbf{p}, u)$ are the associated Hicksian demand functions. The substitution matrix σ would be the matrix of second derivatives of $e(\mathbf{p}, u)$, which, from equations $(1-3)$–$(1-5)$ is

$$\sigma = \begin{pmatrix} -\frac{2}{p_3} & 0 & 2\frac{p_1}{(p_3)^2} \\ 0 & -\frac{2}{p_3} & 2\frac{p_2}{(p_3)^2} \\ 2\frac{p_1}{(p_3)^2} & 2\frac{p_2}{(p_3)^2} & -2\frac{(p_1)^2+(p_2)^2}{(p_3)^3} \end{pmatrix} \qquad (1-6)$$

The matrix σ is negative semi–definite : it has negative entries along the diagonal ; the determinant of the 2–by–2 matrix on the upper left–hand corner is $4/(p_3)^2 > 0$; the determinant of the whole matrix is 0.

So the function $(1-2)$ satisfies all the properties of an expenditure function : it's concave, and it's homogeneous of degree 1 in prices, and the implied Hicksian demands are all positive–valued if the person is on a high enough indifference curve (so that $u > 2\frac{p_1}{p_3}$ and $u > 2\frac{p_2}{p_3}$).

Using Roy's identity, the Marshallian demands can be calculated directly for the "candidate" indirect utility function. When

$$v(\mathbf{p}, y) = \frac{1}{p_1 + p_2}[y + \frac{(p_1)^2 + (p_2)^2}{p_3} - 6p_3]$$

then

$$\frac{\partial v}{\partial y} = \frac{1}{p_1 + p_2} \tag{1 - 7}$$

and

$$\frac{\partial v}{\partial p_1} = -\frac{y}{(p_1 + p_2)^2} + \frac{(p_1)^2 + 2p_1p_2 - (p_2)^2}{(p_3)(p_1 + p_2)^2} + \frac{6p_3}{(p_1 + p_2)^2} \tag{1 - 8}$$

$$\frac{\partial v}{\partial p_2} = -\frac{y}{(p_1 + p_2)^2} + \frac{(p_2)^2 + 2p_1p_2 - (p_1)^2}{(p_3)(p_1 + p_2)^2} + \frac{6p_3}{(p_1 + p_2)^2} \tag{1 - 9}$$

$$\frac{\partial v}{\partial p_3} = -\frac{(p_1)^2 + (p_2)^2}{(p_3)^2(p_1 + p_2)} - \frac{6}{p_1 + p_2} \tag{1 - 10}$$

so that the Marshallian demands ($x_i^M(\mathbf{p}, u) = -v_i/v_y$, where v_i and v_y are the partials of the indirect utility functions) are

$$x_1^M(\mathbf{p}, y) = \frac{1}{p_1 + p_2}[y - \frac{(p_1)^2 - (p_2)^2 + 2p_1p_2}{p_3} - 6p_3] \tag{1 - 11}$$

$$x_2^M(\mathbf{p}, y) = \frac{1}{p_1 + p_2}[y - \frac{(p_2)^2 - (p_1)^2 + 2p_1p_2}{p_3} - 6p_3] \tag{1 - 12}$$

$$x_3^M(\mathbf{p}, y) = 6 + \frac{(p_1)^2 + (p_2)^2}{(p_3)^2} \tag{1 - 13}$$

These Marshallian demand functions satisfy budget balance : from equations $(1 - 11)$ – $(1 - 13)$, $p_1x_1^M(\mathbf{p}, y) + p_2x_2^M(\mathbf{p}, y) + p_3x_3^M(\mathbf{p}, y) = y$, whatever are the prices and income level.

[It's not necessary to derive the expenditure function $(1-2)$ explicitly here. If the Marshallian demand functions $(1-11)$ – $(1-13)$ are computed, using Roy's identity, then the derivatives of the Hicksian demand functions can be calculated from $(1 - 11)$ – $(1 - 13)$ using the Slutsky equation — and that gives a matrix of compensated derivatives which is exactly σ of equation $(1 - 6)$.]

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (20, 5, 5)$ when the price vector was $\mathbf{p} = (5, 20, 10)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	10	10	10	10	10	10
2	5	20	10	20	5	5
3	5	10	20	5	5	10
4	10	5	20	10	5	15

*A*2. One way of finding the violations of the strong and weak axioms of revealed preference is first to construct the matrix, in which the element M_{ij} is the cost of bundle $\mathbf{x}^j$ at prices $\mathbf{p}^i$. Here that matrix is

$$\begin{pmatrix} 300 & 300 & 200 & 300 \\ 350 & 250 & 225 & 300 \\ 350 & 250 & 275 & 400 \\ 350 & 325 & 275 & 425 \end{pmatrix}$$

Using this matrix, the bundle $\mathbf{x}^i$ is directly revealed preferred to the bundle $\mathbf{x}^j$ if $M_{ii} \geq M_{ij}$. For example, row 3 of the matrix has $X_{33} > X_{32}$: that means that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^2$, since bundle $\mathbf{x}^2$ was affordable in period 3 (it cost \$250), and the person instead chose bundle $\mathbf{x}^3$.

The first row shows that bundle $\mathbf{x}^1$ is directly revealed preferred to all of the other bundles, since all of the other three bundles are on or inside the period–1 budget line with equation $10x_1 + 10x_2 + 10x_3 = 300$.

The second row shows that bundle $\mathbf{x}^2$ is directly revealed preferred to bundle 3, but not to the other two bundles.

The third row shows that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle 2, but not to the other two bundles. (It certainly cannot be directly revealed preferred to either bundle 1 or bundle 4, since $\mathbf{x}^3 \leq \mathbf{x}^1$ and $\mathbf{x}^3 \leq \mathbf{x}^4$.)

And the fourth row shows that bundle $\mathbf{x}^4$ is directly revealed preferred to each of the other bundles.

So there are two violations of $WARP$: bundle $\mathbf{x}^1$ compared to bundle $\mathbf{x}^4$ and bundle $\mathbf{x}^2$ compared to bundle $\mathbf{x}^3$.

And there are no additional violations of $SARP$ in this example : bundles $\mathbf{x}^1$ and $\mathbf{x}^4$ are both directly revealed preferred to bundles $\mathbf{x}^2$ and $\mathbf{x}^3$, but bundles $\mathbf{x}^2$ and $\mathbf{x}^3$ are only directly revealed preferred to each other, not to $\mathbf{x}^1$ or $\mathbf{x}^4$.

*Q*3. If a person has a constant coefficient of relative risk aversion equal to 2, what is the probability of winning ρ which must be offered the person — as a function of her initial wealth W — to make her just willing to accept the following bet? The bet : with probability ρ the person wins 1000 dollars, but with probability $1 - \rho$ she loses 1000 dollars.

*A*3. If she has a constant coefficient of relative risk aversion equal to β (with $\beta \neq 1$), then the person's utility–of–wealth fucntion can be written

$$U(W) = \frac{W^{1-\beta}}{1-\beta}$$

So if $\beta = 2$, the person's utility–of–wealth function could be written

$$U(W) = -\frac{1}{W} \tag{3 – 1}$$

[Checking that's right : for the utility function defined in $(3-1)$, $U'(W) = \frac{1}{W^2}$ and $U''(W) = -\frac{2}{W^3}$, so that $R_R = -U''(W)W/U'(W) = 2$.]

If the person accepts the bet described in the question, then her expected utility would be

$$-\rho(W+1000)^{-1} - (1-\rho)(W-1000)^{-1} \tag{3-2}$$

She will be just willing to accept the bet if the expected utility defined in $(3-2)$ is equal to her original utility of $U(W)$ if she chose not to make the bet. So ρ must satisfy the equation

$$-\frac{1}{W} = -\frac{\rho}{W+1000} - \frac{1-\rho}{W-1000} \tag{3-3}$$

Equation $(3-3)$ can be written

$$W^2 - 1000000 = \rho W(W-1000) + (1-\rho)W(W+1000) \tag{3-4}$$

or

$$-1000000 = 1000W((1-\rho)-\rho) \tag{3-5}$$

so that

$$\rho = \frac{W+1000}{2W} \tag{3-6}$$

which is the answer.

Notice that the bet has to be "loaded in her favour" ($\rho > 0.5$) to induce her to take it. But since her preferences exhibit decreasing **absolute** risk aversion (which must be the case if she has a constant coefficient of **relative** risk aversion), the probability of winning ρ falls with W. As W gets very large, $\rho \to 0.5$.

*Q*4. If a person has a constant coefficient of relative risk aversion equal to 2, how much insurance coverage will she want to buy against a loss of L dollars, if the probability of the loss occurring is π, and if the price of one dollar's worth of insurance coverage is p dollars, with $p \geq \pi$?

*A*4. From the answer to question #3 above, the person's preferences can be represented by the utility–of–wealth function

$$U(W) = -\frac{1}{W} \tag{3-1}$$

That means that her expected utility if she chooses to purchase I dollars of insurance coverage will be

$$EU = -\frac{1-\pi}{W-pI} - \frac{\pi}{W-L+(1-p)I} \tag{4-1}$$

The first term in expression $(4-1)$ represents her expected utility from the "good state" : with probability $1-\pi$ she will have no loss, and will have to pay pI in insurance premia. The second term represents the expected utility from the "bad state" : with probability π she will loss L, pay pI in insurance premia, but collect I from the insurance company.

She wants to choose her coverage I so as to maximize her expected utility. So she takes the derivative of expression $(4-1)$ with respect to I and sets it equal to 0. Therefore

$$-\frac{(1-\pi)p}{(W-pI)^2}+\frac{(1-p)\pi}{(W-L+(1-p)I)^2}=0 \tag{4-2}$$

which can be written

$$W-L+(1-p)I=\alpha(W-pI) \tag{4-3}$$

where

$$\alpha\equiv\sqrt{\frac{(1-p)\pi}{p(1-\pi)}}$$

Note that $\alpha\leq 1$, with equality if and only if the insurance premia are actuarially fair ($p=\pi$).

Solving $(4-3)$ for her desired level of coverage I,

$$I=\frac{L-W(1-\alpha)}{1-(1-\alpha)p} \tag{4-4}$$

or

$$I=L\frac{1-\lambda(1-\alpha)}{1-p(1-\alpha)} \tag{4-5}$$

where

$$\lambda\equiv\frac{W}{L}>1$$

is the ratio of her wealth to the size of her loss.

Equation $(4-5)$ shows that the person buys less–than–full coverage if the premia are not actuarially fair. If $\alpha<1$, then the fact that $\lambda>1>p$ shows that $I<L$.

*Q*5. For what values of (x_1,x_2,x_3) does the production function

$$f(x_1,x_2,x_3)=(x_1)^2+A(x_2x_3)^{1/3}$$

exhibit locally increasing returns to scale (where $A>0$ is some constant)?

*A*5. The partial derivatives of the production function are

$$f_1=2x_1 \tag{5-1}$$

$$f_2=\frac{A}{3}(x_2)^{-2/3}(x_3)^{1/3} \tag{5-2}$$

$$f_3=\frac{A}{3}(x_2)^{1/3}(x_3)^{-2/3} \tag{5-3}$$

Therefore

$$f_1x_1+f_2x_2+f_3x_3=2(x_1)^2+\frac{2A}{3}(x_2x_3)^{1/3} \tag{5-4}$$

The production technology exhibits locally increasing returns to scale if and only if $\mu(x_1, x_2, x_3) > 1$, where

$$\mu(x_1, x_2, x_3) \equiv \frac{f_1 x_1 + f_2 x_2 + f_3 x_3}{f(x_1, x_2, x_3)} \tag{5 – 5}$$

From the definition of the production function, and from equation $(5 - 4)$, $\mu(x_1, x_2, x_3) > 1$ if and only if

$$2(x_1)^2 + \frac{2A}{3}(x_2 x_3)^{1/3} > (x_1)^2 + A(x_2 x_3)^{1/3} \tag{5 – 6}$$

which is equivalent to

$$(x_1)^2 > \frac{A}{3}(x_2 x_3)^{1/3} \tag{5 – 7}$$