$Q1$. What is the equation of the short–run supply curve of a firm which has a short–run total cost function with the equation

$$TC(q) = (q - 12)^2 q + 48q + 100$$

where q is the quantity of output produced by the firm?

$A1$. Given the total cost function, the marginal cost function is

$$MC(q) = TC'(q) = 2(q - 12)q + (q - 12)^2 + 48 = 3(q - 12)(q - 4) + 48 \tag{1-1}$$

Setting price equal marginal cost means solving the equation

$$p = 3(q - 12)(q - 4) + 48 \tag{1-2}$$

or

$$3q^2 - 48q + (192 - p) = 0 \tag{1-3}$$

Using the quadratic formula to solve for q,

$$q = 8 + \frac{1}{6}\sqrt{(48)^2 - 4(3)(192 - p)} \tag{1-4}$$

or

$$q = 8 + \frac{\sqrt{12p}}{6} \tag{1-5}$$

which can also be written

$$q = 8 + \sqrt{\frac{p}{3}} \tag{1-6}$$

However, the firm also has the option of shutting down, and producing nothing. It will do so if its revenues are less than its fixed costs of 100.

The condition which ensures that the firm covers its variable costs is that $p \geq AVC$; since it sets $p = MC$, the firm's output level q must be high enough so that $MC \geq AVC$. From the definition of total costs

$$AVC(q) = (q - 12)^2 + 48 \tag{1-7}$$

so that the derivative of the average variable cost function is

$$AVC'(q) = 3(q - 12) \tag{1-8}$$

which reaches a minimum at $q = 12$.

At $q = 12$, equation $(1-2)$ indicates that $MC = 48$, and equation $(1-7)$ indicates that $AVC = 48$.

So if the price is less than 48, the firm is best off producing nothing. The firm's supply curve is the portion of the marginal cost curve which is above its intersection with the average variable cost curve at $p = 148, q = 12$. The equation of the supply curve is

$$q = 0 \qquad \text{if} \qquad p < 48 \tag{1-9}$$

$$q = 8 + \sqrt{\frac{p}{3}} \qquad \text{if} \qquad p > 48 \tag{1-10}$$

(At $p = 48$, the firm has two best options : $q = 0$ or $q = 12$.)

$Q2$. What is the long–run supply curve for the following competitive industry?

Firms differ in their cost of production t. The cost parameter t is distributed uniformly over some interval $[0, T]$.

The total cost of producing q units of output has the following form for a firm with cost parameter t :

$$TC = tq \qquad \text{if} \qquad q \leq 8$$

$$TC = tq + \frac{1}{2}(q-8)^2 \qquad \text{if} \qquad q > 8$$

$A2$. First, note that if the price is p, only firms with $t \leq p$ will choose to produce anything ; the minimum marginal cost of a type–t firm is t.

Second, a firm which does choose to produce, will choose to produce at a level q for which $p = MC$, or

$$t + (q - 8) = p \qquad (2-1)$$

or

$$q = p - t + 8 \qquad (2-2)$$

So if the market price is $p \leq T$, all firms of type p or less will produce, and the total industry output is

$$Q = \int_0^p (p - t + 8)dt \qquad (2-3)$$

or

$$Q = \frac{p^2}{2} + 8p \qquad (2-4)$$

If the price exceeds the maximum possible cost parameter T, then further price increases have no impact on the number of firms : all firms are already in the market. In this case

$$Q = \int_0^T (p - t + 8)dt \qquad (2-5)$$

or

$$Q = pT + [8T - \frac{T^2}{2}] \qquad (2-6)$$

*Q*3. A monopoly faces serves an equal number of two types of customers. The preferences of type–i customers ($i = 1, 2$) can be represented by the utility function

$$U^i(z, x) = z + a_i x - \frac{1}{2}x^2$$

where z is the person's consumption of a numéraire good, competitively supplied at a price of 1 per unit, x is the person's consumption of the good supplied by the monopoly, and a_i is a positive number, with $a_2 > a_1$. Each consumer has the same income M.

The monopoly can provide its good in individual "bundles" : a bundle j contains X_j units of the good, and has a total cost of P_j. Buyers cannot buy individual units of the monopoly's good ; they can't buy multiple bundles ; they either buy one bundle, or they buy nothing.

The monopoly's cost of production is c per unit of output.

What bundles should it offer, and what prices should it charge for each bundle?

*A*3. If a consumer buys a bundle containing X units of the good, and pays a total price P for the bundle, then that leaves her with $M - P$ to spend on the numéraire good, so that her overall utility will be

$$M - P + a_i X - \frac{1}{2}X^2 \qquad (3-1)$$

expression $(3-1)$ means that a type–i person will buy some bundle only if there is a bundle for which

$$a_i X - \frac{1}{2}X^2 - P \geq 0 \qquad (3-2)$$

(Otherwise she is better off spending all her money on the numéraire good, and getting utility of M.) Further, if she has a choice among several different bundles, she will pick the bundle which gives her the highest possible utility : she will pick the bundle which gives her the highest value for $a_i X - \frac{1}{2}X^2 - P$.

So suppose that the monopoly puts two bundles on sale, hoping that buyers of type i choose to buy bundle # i. The monopoly's profit will be proportional to

$$P_1 + P_2 - c(X_1 + X_2) \qquad (3-3)$$

(since there are equal numbers of each type of buyer). But in order to get type–1 buyers to buy "their" bundle 1, the monopoly must price the bundle so that

$$a_1 X_1 - \frac{1}{2}(X_1)^2 - P_1 \geq 0 \qquad (3-4)$$

And in order to get type–2 buyers to buy "their" bundle 2, these buyers must find it more attractive than the other bundle 1, so that

$$a_2 X_2 - \frac{1}{2}(X_2)^2 - P_2 \geq a_2 X_1 - \frac{1}{2}(X_1)^2 - P_1 \qquad (3-5)$$

The monopoly's problem, then, is to choose quantities and prices for the two bundles,

(X_1, P_1, X_2, P_2) to maximize its profit $(3-3)$ subject to the constraints $(3-4)$ that low–value buyers choose to buy a bundle and $(3-5)$ that high–value buyers choose to buy the bunbdle directed to them.

What about the other constraints, that type–2 buyers prefer their bundle to spending all their money on the numéraire good, and that type–1 buyers prefer their bundle to the one directed at type–2 buyers? These constraints, it turns out, will be satisfied at the solution derived below. So adding these two extra constraints will not affect the answer, since these "extra" constraints will not be binding at the monopoly's optimum.

So the Lagrangean is

$$L(X_1, X_2, P_1, P_2) = P_1 + P_2 - c(X_1 + X_2) + \mu[a_1 X_1 - \frac{1}{2}(X_1)^2 - P_1]$$
$$+ \lambda[a_2 X_2 - \frac{1}{2}(X_2)^2 - P_2 - a_2 X_1 + \frac{1}{2}(X_1)^2 + P_1] \qquad (3-6)$$

First–order conditions for maximization of this expression with respect to X_1, X_2, P_1 and P_2 respectively are

$$-c + \mu(a_1 - X_1) - \lambda(a_2 - X_1) = 0 \qquad (3-7)$$

$$-c + \lambda(a_2 - X_2) = 0 \qquad (3-8)$$

$$1 - \mu + \lambda = 0 \qquad (3-9)$$

$$1 - \lambda = 0 \qquad (3-10)$$

Equation $(3-10)$ implies immediately that $\lambda = 1$, and then equation $(3-9)$ implies that $\mu = 2$. Plugging $\lambda = 1$ into equation $(3-8)$ yields

$$a_2 - X_2 = c \qquad (3-11)$$

Equation $(3-11)$ says that the monopoly provides an efficient quantity to the high–taste customers : their marginal willingness to pay for a little more of the good, $a_2 - X_2$ equals the marginal cost of producing a little more.

Plugging $\lambda = 1$ and $\mu = 2$ into equation $(3-7)$,

$$a_1 - X_1 = c + (a_2 - a_1) \qquad (3-12)$$

so that the low–taste customers do not get an efficient quantity of the good : their marginal willingness to pay $a_1 - X_1$ is greater than the marginal cost of supplying a little more of the good. Here the monopoly provides an inefficiently low quantity of the good to the low–taste buyers in order to increase its profits on bundle 2. By making the low–quantity, low–price bundle less attractive, they are able to charge a higher price for the high–quantity bundle, and still get the high–taste customers to choose to buy it.

The monopoly extracts all the rent it can from the low–taste buyers : constraint $(3-4)$ holds as an equality (since the associated Lagrange multiplier μ is strictly positive).

There's no worry about low–taste buyers wanting to buy the "wrong" bundle : constraint $(3-5)$ holds as an equality (since $\lambda > 0$), and that means that

$$a_1X_2 - \frac{1}{2}(X_2)^2 - P_2 - a_1X_1 + \frac{1}{2}(X_1)^2 + P_1 < a_2X_2 - \frac{1}{2}(X_2)^2 - P_2 - a_2X_1 + \frac{1}{2}(X_1)^2 + P_1 = 0 \quad (3-13)$$

And high–taste buyers do not have all their consumer surplus extracted by the monopoly : equations $(3-4)$ and $(3-3)$, and the fact that $a_2 > a_1$ ensure that

$$a_2X - \frac{1}{2}(X_2)^2 - P_2 \geq 0 \quad (3-14)$$

[In this situation, there are no quantity discounts in the bundling. Quite the opposite. Here the price per unit P_i/X_i is higher for the bigger bundle, as the monopoly exploits the higher willingness–to–pay of group 2. That's why the constraint of "one bundle to a customer" may be necessary, since otherwise the high–taste customers might be better off buying 2 of the smaller bundles X_1 instead of one of bundle X_2.]

Q4. What is the symmetric Bertrand equilibrium in a world in which all consumers have exogenous incomes y, all have CES preferences

$$U(\mathbf{x}) = ((x_1)^\rho + (x_2)^\rho + \cdots + (x_n)^\rho)^{1/\rho}$$

with $0 < \rho < 1$, in which each good is produced by a single producer with a constant marginal production cost c?

A4. (As in Jehle and Reny), a consumer's demand for good i is

$$x_i = \frac{(p_i)^{r-1}}{\sum_{j=1}^n (p_j)^r} y \tag{4 – 1}$$

where

$$r = \frac{\rho}{\rho - 1} < 0$$

when the consumers have identical CES preferences.

The profit of a firm is $p_i x_i - c x_i$ if it has constant marginal costs, so that equation $(4-1)$ implies that each firm maximizes

$$\frac{p_i^r}{\sum_{j=1}^n p_j^r} y - c \frac{p_i^{r-1}}{\sum_{j=1}^n p_j^r} y \tag{4 – 2}$$

with respect to p_i if it behaves as a Bertrand (price–setting) oligopolist.

Maximizing expression $(4-2)$ with respect to p_i — and recognizing that p_i is one of the elements in the summation in the denominator, as well as in the numerator — the first–order conditions for profit maximization by firm i are

$$(rp_i^{r-1} - c(r-1)p_i^{r-2})(\sum_{j=1}^n p_j^r) - rp_i^{r-1}(p_i^r - cp^{r-1}) = 0 \tag{4 – 3}$$

In a symmetric equilibrium $p_1 = p_2 = \cdots = p_n = p$, so $\sum_{j=1}^n p_j^r = np^r$, and expression $(4-3)$ can be written

$$p(n-1)r - c[n(r-1) - r] = 0 \tag{4 – 4}$$

so that

$$p = \frac{n(1-r) + r}{-(n-1)r} c \tag{4 – 5}$$

The fact that $r < 0$ means that expression $(4-5)$ is positive, and exceeds the marginal cost c.

If goods are very good substitutes $\rho \to 1$, then the equilibrium price approaches c, as in the "standard" Bertrand model of price–setting when firms' products are perfect substitutes. And taking derivatives of $(4-5)$ with respect to r and to n, the equilibrium price decreases when the goods are better substitutes (r falls), or when there are more firms (n increases).

*Q*5. Suppose that a firm in a Cournot duopoly can raise its rival's marginal cost of production by some investment. This investment is costly, and has no effect on the firm's own marginal cost of production.

In particular, the total costs incurred by firm 1, if it produces q_1 units of output, are $c_1 q_1 + \beta(c_2)^2$ — and firm 2's costs are $c_2 q_2 + \beta(c_1)^2$, where β is a positive constant (with $\beta > 1/3$).

The decisions proceed as follows. First each firm chooses how much to invest in increasing the other firm's costs. That is, initially firm #1 chooses c_2, and incurs a cost of $\beta(c_2)^2$ and firm 2 chooses c_1 and incurs a cost of $\beta(c_1)^2$.

Each firm then observes what its own costs are, and the firms play a Cournot game (choosing output levels, given the marginal costs c_1 and c_2 which have already been determined). Aggregate demand for the firms' homogeneous output obeys the equation $p = a - Q$ for some positive a, where p is the market price, and Q the aggregate output in the industry.

(*i*) In a symmetric equilibrium, what cost levels c_2 and c_1 are chosen by the firms in the initial stage of the game (when they choose each others' costs, simultaneously and non–cooperatively)?

(*ii*) How do firms' equilibrium profits vary with the cost β of investment in cost–raising?

*A*5. Start first at the last stage of this process, when the firms play a Cournot game, after costs have already been determined. Firm 1 chooses its output level q_1 to maximize

$$\pi_1 = (a - q_1 - q_2)q_1 - c_1 q_1 - \beta(c_2)^2 \qquad (5-1)$$

This maximization has a first–order condition

$$q_1 = \frac{a - c_1}{2} - \frac{q_2}{2} \qquad (5-2)$$

Similarly, firm 2's optimal choice of output is

$$q_2 = \frac{a - c_2}{2} - \frac{q_1}{2} \qquad (5-3)$$

Solving $(5-2)$ and $(5-3)$ simultaneously yields the Cournot output levels

$$q_1 = \frac{a}{3} - \frac{2c_1 - c_2}{3} \qquad (5-4)$$

$$q_2 = \frac{a}{3} - \frac{2c_2 - c_1}{3} \qquad (5-5)$$

The equilibrium price is

$$p = a - q_1 - q_2 = \frac{a + c_1 + c_2}{3} \qquad (5-6)$$

From $(5-6)$, firm 1's profit in the Cournot equilibrium is

$$\pi_1 = (p - c_1)q_1 - \beta(c_2)^2 = \frac{(a - 2c_1 + c_2)^2}{9} - \beta(c_2)^2 \qquad (5-7)$$

when the firms' marginal costs are c_1 and c_2.

In the initial stage of the game, firm 1 chooses its rival's cost varaiable c_2 so as to maximize its own profit, defined by $(5-7)$. Differentiating $(5-7)$ with respect to c_2,

$$\frac{\partial \pi_1}{\partial c_2} = \frac{2}{9}(a - 2c_1 + c_2) - 2\beta c_2 \qquad (5-8)$$

[The second–order conditions for a maximum are satisfied here if $\beta > 1/9$.] So firm 1's best choice of c_2, given the Cournot competition which will follow in the second stage, is the value of c_2 for which $\partial \pi_1 / \partial c_2 = 0$, or

$$c_2 = \frac{a - 2c_1}{9\beta - 1} \qquad (5-9)$$

Similarly, firm 2 invests in technology to increase its rival's costs to the point at which

$$c_1 = \frac{a - 2c_2}{9\beta - 1} \qquad (5-10)$$

If both firms pick these cost variables simultaneously in the first stage, then the equilibrium level of costs are the levels which satisfy $(5-9)$ and $(5-10)$, or

$$c_1 = c_2 = \frac{a}{9\beta + 1} \qquad (5-11)$$

which is the answer to part (i) of the question.

Plugging in for c_1 and c_2 into the definition of profits $(5-6)$, firm 1's equilibrium profits are

$$\pi_1 = \frac{a^2}{9} \frac{\beta^2}{(9\beta + 1)^2} \qquad (5-12)$$

Expression $(5-12)$ is an increasing function of β. So the more costly it is for firm's to make things difficult for their rival, the better off firms will be in equilibrium. The temptation to gain a strategic advantage in the (subsequent) Cournot game makes each firm worse off, as they "waste" money trying to gain an advantage (or to give their rival less of an advantage) in the Cournot competition.