GS/ECON 5010 section "B"' Answers to Assignment 4 November 2013

*Q*1. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of E_1 units of good 1 and E_2 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(\mathbf{x}^1) = 100 - (x_1^1)^{-\beta+1} - (x_2^1)^{-\beta+1}$$

$$u^2(\mathbf{x}^2) = 100 - a(x_1^2)^{-\beta+1} - (x_2^2)^{-\beta+1}$$

where $a > 1$, $\beta > 1$, and x_j^i is person i's consumption of good j? [The superscripts in the definition of u^2 are the person's name, "2", not "squared".]

*A*1. An allocation $(\mathbf{x}^1, \mathbf{x}^2)$, with $\mathbf{x}^i >> 0$, will be efficient if and only if the two people's marginal rates of substitution are equal, or

$$\frac{u_1^1}{u_2^1} = \frac{u_1^2}{u_2^2} \tag{1-1}$$

where u_j^i is person i's marginal utility from good j.

Here, that means that

$$[\frac{(x_2^1)}{(x_1^1)}]^\beta = a[\frac{(x_2^2)}{(x_1^2)}]^\beta \tag{1-2}$$

or

$$\frac{x_2^1}{x_1^1} = A\frac{x_2^2}{x_1^2} \tag{1-3}$$

where

$$A \equiv a^{\frac{1}{\beta}} > 1$$

Since $x_2^2 = E_2 - x_2^1$ and $x_1^2 = E_1 - x_1^1$, equation $(1-3)$ can be written

$$\frac{x_2^1}{x_1^1} = A\frac{E_2 - x_2^1}{(E_1 - x_1^1)} \tag{1-4}$$

which defines x_1^2 as a function of x_1^1.

Equation $(1-4)$ can be written

$$x_2^1 = \frac{AE_2x_1^1}{E_1 + (A-1)x_1^1} \tag{1-5}$$

which defines an upward–sloping curve in the Edgeworth box. Equation $(1-5)$ implies that $x_2^1 = 0$ when $x_1^1 = 0$, and that $x_2^1 = E_2$ when $x_1^1 = E_1$. So it goes through the corners of the Edgeworth box. That means that there are no Pareto optima to worry about in which consumption of some good by some person is zero.

Differentiating the right side of $(1-5)$ with respect to x_1^1, the slope of the contract curve is

$$\frac{dx_2^1}{dx_1^1} = \frac{AE_1E_2}{(E_1 + (A-1)x_1^1)^2} \qquad (1-6)$$

which is positive. When $A > 1$, the derivative of the right–hand side of equation $(1-6)$ with respect to x_1^1 is negative : so the slope of the contract curve falls as we go from the bottom left corner of the Edgeworth box to the top right corner.

So the contract curve is everywhere (except at the corners) above the diagonal of the box, reflecting the difference in preferences : person two has a relatively stronger preference for good 1, so that $x_1^1/x_2^1 < x_1^2/x_2^2$ at every efficient allocation in the interior of the Edgeworth box.

(Equation $(1-6)$ also shows that the slope of the contract curve is $AE_2/E_1 > E_2/E_1$ when $x_1^1 = x_2^1 = 0$, so that the contract curve moves up above the diagonal as we move right from the bottom left corner of the Edgeworth box.)

(The fact that the contract curve is above the diagonal can be derived directly as well. Equation $(1-5)$ says that $x_2^1/x_1^1 > E_2/E_1$ if and only if

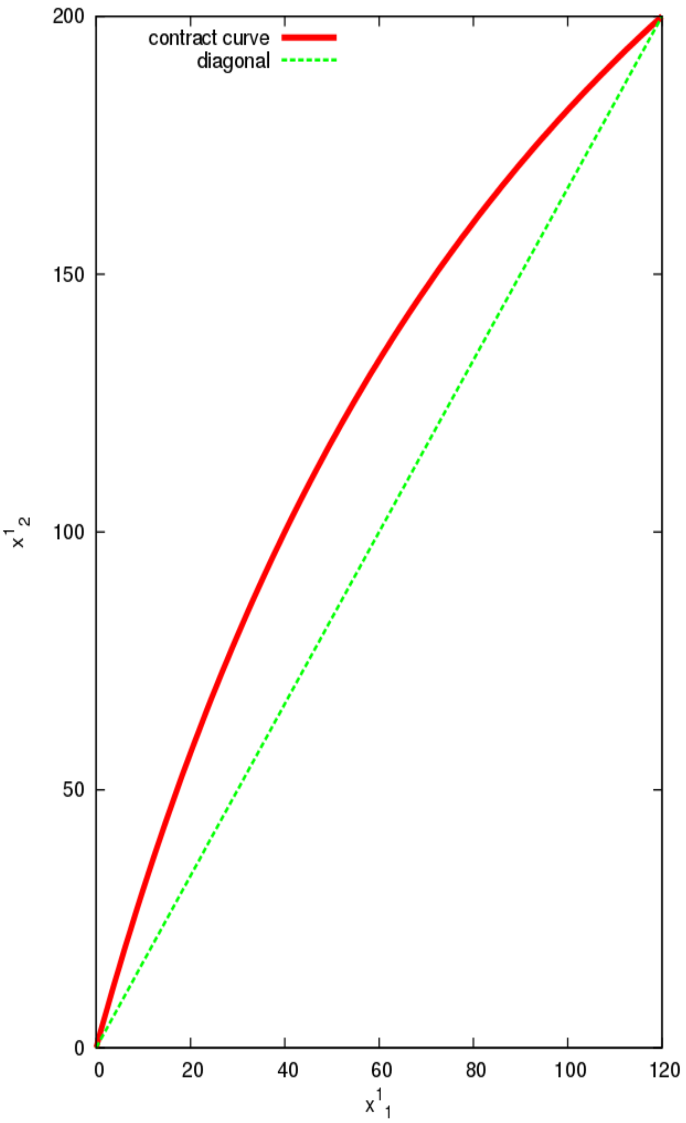
$$\frac{AE_2}{E_1 + (A-1)x_1^1} > \frac{E_2}{E_1} \qquad (1-7)$$

which is the same thing as

$$(A-1)E_1 > (A-1)x_1^1 \qquad (1-8)$$

which must be true if $a > 1$ and $x_1^1 < E_1$.)

(The accompanying figure shows the contract curve when $a = 4$, $\beta = 2$, $E_1 = 120$ and $E_2 = 200$.)



contract curve ($a = 4$, $\beta = 2$, $E_1 = 120$ and $E_2 = 200$)

Q2. Find 3 allocations in the core of the following 3–person, 2–good economy.

Person i's preferences can be represented by the utility function $u^i(x_1^i, x_2^i)$, where

$$u^1(x_1^1, x_2^1) = x_1^1$$

$$u^2(x_1^2, x_2^2) = x_2^2$$

$$u^3(x_1^3, x_2^3) = x_1^3 x_2^3$$

and the endowment vectors of the three people are $\mathbf{e}^1 = \mathbf{e}^2 = \mathbf{e}^3 = (1, 1)$.

A2. Since person 1 likes only good 1, and person 2 likes only good 2, then any Pareto optimal llocation must be of the form

$$\mathbf{x}^1 = (a, 0) \tag{2-1}$$

$$\mathbf{x}^2 = (0, b) \tag{2-2}$$

$$\mathbf{x}^3 = (3-a, 3-b) \tag{2-3}$$

for some numbers a and b, with $0 < a, b < 3$.

Since each person can block an allocation by consuming her own endowment, it must be true that $a \geq 1$, and that $b \geq 1$. (Otherwise person 1 or 2 would block the allocation.)

It also must be true that either $a \leq 2$ or $b \leq 2$. Why? Otherwise (if both a and b were greater than 2), then person 3 would have a consumption bundle $\mathbf{x}^3 = (3-a, 3-b) << (1, 1)$, so that she would block the allocation by consuming her own endowment vector $(1, 1)$.

What if a and b were **both** less than 2? Then a coalition of person 1 and person 2 would block the allocation, since a coalition of 1 and 2 would get the consumption bundles $\mathbf{y}^1 = (2, 0), \mathbf{y}^2 = (0, 2)$ by splitting up their own total endowment $\mathbf{e}^1 + \mathbf{e}^2 = (2, 2)$ efficiently.

So the only possible allocations in the core must have $a \geq 2 \geq b$, or $b \geq 2 \geq a$.

Consider now an allocation in which $a > 2 > b$.

Equation $(2-3)$ says that person 3's utility in this allocation would be

$$u^3 = (3-a)(3-b) = 9 - 3a - 3b + ab \tag{2-4}$$

If person 3 were to form a coalition to block this allocation, she should try forming one with person 2, since she only has to offer person 2 a utility level of $b < 2$ to get him to join the blocking coalition. (To get person 1 to join a 2–person blocking coalition, person 3 would have to offer her a utility level of $a > 2$, which leaves less for person 3.)

So if person 2 and person 3 formed a coalition, and the coalition allocated b units of good 2 to person 2, that would leave person 3 with the remainder of the coalition's endowment, namely

$$\mathbf{y}^3 = (2, 2-b) \tag{2-5}$$

giving her utility of

$$\tilde{u}^3 = 2(2-b) = 4 - 2b \tag{2-6}$$

This coalition makes person 3 better off (than the proposed allocation $(a, 0), (0, b), (3 - a, 3 - b)$) only if $\tilde{u}^3 > u^3$, or (from equations $(2-4)$ and $(2-6)$)

$$9 - 3a - 3b + ab < 4 - 2b \qquad (2-7)$$

which is equivalent to

$$5 - 3a - b + ab < 0 \qquad (2-8)$$

If condition $(2-8)$ holds, then the allocation is **not** in the core, since it can be blocked by a coalition of person 2 and person 3. So an allocation, with $a > 2 > b$, is in the core only if $(2-8)$ does **not** hold. As well, the original allocation must be better for person 3 than her original endowment. Her original endowment gives her a utility of $(1)(1) = 1$, so that she will block the proposed allocation by consuming her own endowment if a and b are so large that

$$u^3 = 9 - 3a - 3b + ab < 1 \qquad (2-9)$$

So an allocation in which $a > 2 > b > 1$ will be in the core if and only if conditions $(2-8)$ and $(2-9)$ do **not** hold.

There are such allocations, for example $(a, b) = (2.2, 1.4)$, or $(a, b) = (2.1, 1.5)$.

The last few paragraphs started with the assumption that $a > 2 > b$. If instead $b > 2 > a$, then person 3 would be looking to form a coalition with person 1 in order to block the allocation. She will able to do so if and only

$$5 - 3b - a + ab < 0 \qquad (2-10)$$

(which is just $(2-8)$ with the a's and b's transposed). Again, a and b must be small enough such that $(2-9)$ does not hold, or person 3 could block the allocation on her own. So $(a, b) = (1.4, 2.2)$ or $(a, b) = (1.5, 2.1)$ would lead to allocations in the core.

And the allocation defined by $a = b = 2$ is also in the core : that's the competitive equilibrium allocation for this economy.

In summary, any allocation satisfying conditions $(2-1)-(2-3)$ will be in the core, if any of the following three sets of conditions hold :

(i) $a = b = 2$

(ii) $a > 2 > b$, with $5 - 3a - b + ab \leq 0$ and $9 - 3a - 3b + ab \geq 1$

(iii) $b > 2 > a$, with $5 - a - 3b + ab \leq 0$ and $9 - 3a - 3b + ab \geq 1$

*Q*3. Find the competitive equilibrium to a 2–good, 3–million–person economy, in which each person's endowment is $\mathbf{e}^i = (1, 1)$, and which 1 million people have preferences like person 1 in the previous question (# 2), 1 million people have preferences like person 2 in the previous question, and 1 million people have preferences like person 3 in the previous question. [That is, find a competitive equilibrium to an economy which is the economy of question #2 cloned one million times.]

*A*3. Suppose that good 1 is the numéraire, and that the relative price of good 2 is denoted p, so that the price vector is $\mathbf{p} = (1, p)$. Then each person's income will be $m^i = 1 + p$, since each person's endowment is $(1, 1)$.

Person 1 cares only about good 1, and so will spend all her income on good 1. Therefore, her excess demand for good 1 — when the price of good 1 is 1 — is her income, minus her endowment of good 1 :

$$z_1^1(1, p) = (1 + p) - 1 = p \tag{3 – 1}$$

Person 2 will never want to buy any of good 1, so that her excess demand for good 1 is

$$z_1^2(1, p) = -1 \tag{3 – 2}$$

Person 3 has Cobb–Douglas preferences, so that her demand for good 1 is $m^3/2$, when m^3 is her income, and when the price of good 1 is 1. Since here $m^3 = 1 + p$, her excess demand therefore is

$$z_1^3(1, p) = \frac{1 + p}{2} - 1 = \frac{p - 1}{2} \tag{3 – 3}$$

In equilibrium, the aggregate excess demand for good 3 is the sum of all people's excess demands, or

$$Z_1(1, p) = 1000000[p - 1 + \frac{p - 1}{2}] \tag{3 – 4}$$

Therefore, the market for good 1 clears when $p = 1$. Since only relative prices matter, the Walrasian equilibrium price vectors for this economy are any price vectors $p = (\zeta, \zeta)$, for any $\zeta > 0$.

Returning to the case in which good 1 is numéraire, so that $\zeta = 1$, each person's income is 2, so that person 1's demand for good 1 equals 2, person 2's demand for good 1 is 0, and person 3's demand is 1. The Walrasian equilibrium allocation is : $\mathbf{x}^1 = (2, 0)$, $\mathbf{x}^2 = (0, 2)$, $\mathbf{x}^3 = (1, 1)$.

Q4. Write down the strategic form for the game described below, and find all the Nash equilibria to it.

The two players in the game are governments of two regions. Each government is trying to attract a firm, which will build a factory in one of the regions. (The factory's owner is **not** a player in this game.)

The two governments are choosing (simultaneously) whether or not to offer a tax exemption to the factory. It is common knowledge that the factory will locate in the region offering the tax exemption, if only one region chooses to offer an exemption. If neither region offers an exemption — or if both regions offer an exemption — the factory owner will flip a coin, and locate in either region with probability 0.5.

Each government places a value of 100 million dollars on having the factory in its region. A government will also collect T million dollars in taxes from the factory — if the factory locates in the region, and if the region has **not** offered a tax exemption. If the region offers a tax exemption, it collects no tax revenue from the factory.

Governments are risk neutral, and seek to maximize their expected returns, from having a factory in their region and from any taxes collected from the factory.

A4. If the returns are denominated in millions of dollars, a government's payoff is $100 + T$ if it gets the factory and collects the taxes, 100 if it gets the factory but has given a tax exemption to the factory, and 0 otherwise. Given that the factory will locate in a region with probability 0.5 if the two regions choose identical policies, the strategic form of the game can be written

	e	ne
E	$(50, 50)$	$(100, 0)$
NE	$(0, 100)$	$(50 + T/2, 50 + T/2)$

where e and ne denote the strategies "offer an exemption" and "don't offer an exemption".

If $T < 100$, then each player has a strictly dominant strategy, which is to offer an exemption. The unique Nash equilibrium is for each government to offer an exemption.

In the razor's edge case $T = 100$, offering an exemption is still a weakly dominant strategy. But there is another Nash equilibrium, which involves each government playing a weakly dominated strategy, namely not to offer an exemption. (And there are no mixed strategy equilibria.)

If $T > 100$, then there are no dominant strategies. There are two pure–strategy Nash equilibria : both governments offer an exemption, and neither government offers an exemption. There is also a mixed strategy equilibrium in which each government offers an exemption with probability $1 - 100/T$, and refuses to offer an exemption with probability $100/T$. [The expected payoff in this mixed strategy equilibrium is $50 + 5000/T$, which decreases with the amount of tax revenue which can be collected.]

$Q5$. Find all the pure–strategy Nash equilibria to the game described in the previous question, if there now are $N > 2$ different regions (and still only one factory), so that, if $I > 0$ regions offer a tax exemption, each of the I regions gets the factory with probability $1/I$, and if none of the regions offers a tax exemption then each of them gets the factory with probability $1/N$.

$A5$. Suppose that I other regions' governments have chosen to offer a tax exemption (with $0 < I < N$). Then my region will get the factory with probability $1/(I+1)$ if my government offers an exemption, and will get the factory with probability zero if we don't offer an exemption. This will be the case if $I > 0$: as long as at least one other region has offered an exemption, my region's government's best response if to offer an exemption as well.

So there are two implications from the previous paragraph. First, there will always be a Nash equilibrium in which all N regions offer exemptions. Second, there cannot be a pure–strategy Nash equilibrium in which I regions offer an exemption, if $1 < I < N$. [Why not? Because if at least one other region offers an exemption, every other region's best response is to offer an exemption themselves.]

Other than the equilibrium in which all N regions offer an exemption, there is only one other pure–strategy Nash equilibrium possible. That other Nash equilibrium is for **none** of the regions to offer an exemption. There will be such a Nash equilibrium if (and only if) region 1's best strategy is not to offer an exemption, even when none of the other $I-1$ regions offer exemptions.

So if regions $2, 3, \ldots, N$ choose not to offer an exemption, then region 1's payoff will be 100 if its government choose to offer an exemption : being the only region to offer an exemption, they get the factory for certain. On the other hand, if they too choose not to offer an exemption, then their payoff will be $(100+T)/N$, since there is still a $1/N$ chance they get the factory, if they (and all of the other regions) choose not to offer an exemption.

So there will be another Nash equilibrium if

$$100 \leq \frac{100+T}{N} \qquad (5-1)$$

or

$$T \geq 100(N-1) \qquad (5-2)$$

(which generalizes the condition for multiple equilibria from the previous question). If condition $(5-2)$ does not hold, then offering an exemption is a dominant strategy, and there is a unique Nash equilibrium. If condition $(5-2)$ holds, then there will be another pure–strategy Nash equilibrium in which no region offers an exemption. (And if inequality $(5-2)$ is strict, then there will also be at least one mixed–strategy Nash equilibrium.)