*Q*1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

There are 3 different commodities. The person finds bundle $\mathbf{x^1}$ at least as good as bundle $\mathbf{x}^2$ if and only if : the minimum of x_1^1 and $\max(x_2^1, x_3^1)$ is at least as big as the minimum of x_1^2 and $\max(x_2^2, x_3^2)$. (Where $\max(a, b)$ denotes the maximum of a and b.)

*A*1. The preferences are strictly monotonic. If $\mathbf{x}^1 \geq \mathbf{x}^2$, then $x_1^1 \geq x_1^2$, and $x_1^2 \geq x_2^2$ and $x_1^3 \geq x_2^3$, so that $\max(x_1^2, x_1^3) \geq \max(x_2^2, x_2^3)$, which means that $\mathbf{x}^1$ must be considered at least as good as bundle 2. Similarly, if $\mathbf{x}^1 >> \mathbf{x}^2$, then $\min(x_1^1, \max(x_1^2, x_1^3)) > \min(x_2^1, \max(x_2^2, x_2^3))$ so that $\mathbf{x}^1$ would be preferred strictly to $\mathbf{x}^2$.

But the preferences are **not** convex. The "maximum" function is not quasi–concave, although the "minimum" function is. Mixing up the maximum and the minimum, as in this example, does not ensure that **every** "at least as good" set is convex.

To prove that preferences are **not** convex, the easiest thing to do is to provide a counter–example. A single example of an "at least as good" set which is not convex demonstrates that the preferences are not convex. (And the presence of the "maximum" operator in the question should suggest that preferences may not be convex.)

As an example, this person is indifferent between the bundles $\mathbf{x}^1 \equiv (5, 1, 3)$ and $\mathbf{x}^2 \equiv (5, 3, 1)$: in each case $\min(x_1^i, \max(x_2^i, x_3^i)) = 3$, for $i = 1, 2$. But the the bundle which is halfway between $\mathbf{x}^1$ and $\mathbf{x}^2$ is $\mathbf{x}^3 \equiv (5, 2, 2)$, and this person prefers $\mathbf{x}^1$ and $\mathbf{x}^2$ strictly to $\mathbf{x}^3$ here.

*Q*2. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1, x_2, x_3) = \frac{x_1}{x_1 + x_2} - \frac{1}{x_3}$$

*A*2. The preferences are **not** strictly monotonic, because the utility function is strictly decreasing in the quantity x_2 of good #2 : the derivative of $U(x_1, x_2, x_3)$ with respect to x_2 is $-\frac{x_1}{(x_1+x_2)^2} < 0$.

The preferences are also **not** convex. One indication why this might be the case is that the second derivative

$$\frac{\partial^2 U}{\partial (x_2)^2} = \frac{2x_1}{(x_1 + x_2)^3} > 0$$

is positive.

To show that preferences are not convex, it is sufficient to provide a single counter–example, a pair of consumption bundles $\mathbf{x}^1$ and $\mathbf{x}^2$ such that $U(\mathbf{x}^1) = U(\mathbf{x}^2)$ but $U(\mathbf{x}^3) < U(\mathbf{x}^1)$ for some $\mathbf{x}^3 = t\mathbf{x}^1 + (1 - t)\mathbf{x}^2$ with $0 < t < 1$.

For example, $U(1,1,1) = -\frac{1}{2}$ and $U(1,0,\frac{2}{3}) = 1 - \frac{3}{2} = -\frac{1}{2}$. But the consumption bundle which is halfway between $(1,1,1)$ and $(1,0,\frac{2}{3})$ is the bundle $(1,\frac{1}{2},\frac{5}{6})$, and

$$U(1,\frac{1}{2},\frac{5}{6}) = \frac{2}{3} - \frac{6}{5} = -\frac{8}{15} < -\frac{1}{2}$$

Another way of providing a counter–example is to look at indifference curves in 2 dimensions. Consider, for example, the set of (x_2, x_3) combinations which yield a utility level of $U = -0.5$ when $x_1 = 1$. This curve has the equation

$$x_3 = \frac{1 + x_2}{1 + (1.5)x_2} \qquad (2-1)$$

If equation $(2-1)$ is graphed, with x_2 on the horizontal axis and x_3 on the vertical, it's an upward sloping curve. And the "better–than" set — the (x_2, x_3) combinations which are above and to the left of the curve is not convex. [And having "better than" sets in 2 dimensions convex is a necessary — but not sufficient — conditions for the preferences to be convex when the number of commodities is greater than 2.]

*Q*3. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1(x_2 + x_3)^2$$

*A*3. The term $x_2 + x_3$ in the utility function suggests (I hope) that preferences here are not strictly convex, and there are going to be some flat parts to the indifference surfaces. This means that there may not be an interior solution, with $(x_1, x_2, x_3) >> 0$, to the consumer's problem.

The first–order conditions from maximization of $u(x_1, x_2, x_3)$ subject to the budget constraint confirm this suggestion. The first–order conditions are

$$(x_2 + x_3)^2 = \lambda p_1 \qquad (3-1)$$

$$2(x_2 + x_3)x_1 = \lambda p_2 \qquad (3-2)$$

$$2(x_2 + x_3)x_1 = \lambda p_3 \qquad (3-3)$$

Equations $(3-2)$ and $(3-3)$ **cannot** both be satisfied at the same time — unless $p_2 = p_3$. If $p_3 > p_2$, for example, then if equation $(3-2)$ holds, then it must be true that $\frac{\partial u}{\partial x_3} - \lambda p_3 < 0$: utility will increase by lowering consumption of good 3 (spending the extra money on the other two goods).

So if $p_3 > p_2$, the consumer's optimum must have $x_3 = 0$, and if $p_2 > p_3$, the optimum must have $x_2 = 0$.

Another way of seeing the same thing : if $p_3 > p_2$ and $x_3 > 0$, then decreasing x_3 by some small quantity ϵ, and spending the money saved, $p_3\epsilon$, must increase $x_2 + x_3$ by $\frac{(p_3 - p_2)}{p_2}\epsilon > 0$, and thus increase utility.

Now if $x_3 = 0$, the consumer is maximizing $x_1(x_2)^2$ subject to a budget constraint. That's a Cobb–Douglas utility function, with Marshallian demands $x_1 = \frac{y}{3p_1}, x_2 = \frac{2y}{3p_2}$.

If $p_2 > p_3$, then x_2 should equal zero, so that the consumer again is maximizing a Cobb–Douglas utility function $x_1(x_3)^2$.

So the consumer's Marshallian demand function for good #1 is

$$x_1^M(\mathbf{p}, y) = \frac{y}{3p_1} \tag{3 – 4}$$

Her Marshallian demand function for good #2 is

$$x_2^M(\mathbf{p}, y) = \frac{2y}{3p} \quad \text{if} \quad p_2 < p_3 \tag{3 – 5a}$$

$$x_2^M(\mathbf{p}, y) = 0 \quad \text{if} \quad p_2 > p_3 \tag{3 – 5b}$$

and, similarly,

$$x_3^M(\mathbf{p}, y) = 0 \quad \text{if} \quad p_2 < p_3 \tag{3 – 6a}$$

$$x_3^M(\mathbf{p}, y) = \frac{2y}{3p} \quad \text{if} \quad p_2 > p_3 \tag{3 – 6b}$$

If $p_2 = p_3$, the fact that goods #2 and #3 are perfect substitutes means that any combination of x_2 and x_3, with the same sum $x_2 + x_3$, yields the same utility. So when $p_2 = p_3$, the consumer's optimization does not have a unique solution ; any $(x_1, x_2, x_3) \geq 0$ satisfying $(3 - 4)$ and $(3 - 7)$ (below) will be optimal.

$$x_2^M(\mathbf{p}, y) + x_3^M(\mathbf{p}, y) = \frac{2y}{3p_2} = \frac{2y}{3p_3} \tag{3 – 7}$$

*Q*4. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2) = \frac{x_1 x_2}{x_2 + 1}$$

*A*4. This utility form represents the same preferences as the function $U(x_1, x_2) \equiv \log[u(x_1, x_2]$, or

$$U(x_1, x_2) = \log(x_1) + \log(x_2) - \log(x_2 + 1) \tag{4 – 1}$$

Using $U(x_1, x_2)$, the first–order conditions to the consumer's utility maximization problem are

$$\frac{1}{x_1} = \lambda p_1 \tag{4 – 2}$$

$$\frac{1}{x_2} - \frac{1}{x_2 + 1} = \frac{1}{x_2(x_2 + 1)} = \lambda p_2 \tag{4 – 3}$$

Since $(4-2)$ implies that $\lambda = \frac{1}{p_1(1+x_1)}$, equation $(4-3)$ can be written

$$\frac{x_1}{x_2(x_2+1)} = \frac{p_2}{p_1} \tag{4-4}$$

or

$$x_1 = \frac{p_2}{p_1}(x_2(x_2+1)) \tag{4-5}$$

Substitution from $(4-5)$ for x_1 into the budget constraint $p_1x_1 + p_2x_2 = y$ yields

$$p_2x_2(x_2+1) + p_2x_2 = y \tag{4-6}$$

or

$$p_2(x_2)^2 + 2(p_2x_2) - y = 0 \tag{4-7}$$

Equation $(4-7)$ is a quadratic function of x_2, and can be solved, using the quadratic formula, as

$$x_2^M(\mathbf{p}, y) = \frac{\sqrt{p_2+y}}{\sqrt{p_2}} - 1 \tag{4-8}$$

which implies (from equation $(4-5)$) that the Marshallian demand function for good #1 is

$$x_1^M(\mathbf{p}, y) = \frac{p_2 + y - \sqrt{p_2}\sqrt{p_2+y}}{p_1} \tag{4-9}$$

*Q*5. Find the expenditure function, Hicksian demand functions and indirect utility function for the preferences of question #4 above.

*A*5. Staring with the indirect utility function, substitution from $(4-9)$ and $(4-8)$ into the utility function defined in question 4 implies that

$$v(p_1, p_2, y) = \frac{(p_2 + y - \sqrt{p_2}\sqrt{p_2+y})(\sqrt{p_2+y} - \sqrt{p_2})}{p_1\sqrt{p_2+y}} \tag{5-1}$$

which can be simplified into

$$v(p_1, p_2, y) = \frac{(\sqrt{p_2+y} - \sqrt{p_2})^2}{p_1} \tag{5-2}$$

To get the Hicksian demand functions, we can start with equation $(4-5)$ from the answer to question $4:$ this equation is a consequence of the condition $MRS_{12} = p_1/p_2$, (where MRS denotes the marginal rate of substitution between goods), which must hold for both Hicksian and Marshallian demands, and must hold for any utility function representing the preferences of question #4.

Substitution from $(4-5)$ into the expression for the utility function, $u = \frac{x_1x_2}{x_2+1}$ yields

$$u = \frac{p_2}{p_1}(x_2)^2 \tag{5-3}$$

or

$$x_2^H(p_1, p_2, u) = \sqrt{u}\frac{\sqrt{p_1}}{\sqrt{p_2}} \tag{5 − 4}$$

which is the Hicksian demand function for good #2. Substituting from $(5-4)$ into $(4-5)$,

$$x_1^H(p_1, p_2, u) = u + \sqrt{u}\frac{\sqrt{p_2}}{\sqrt{p_1}} \tag{5 − 5}$$

which means that the expenditure function,

$$e(p_1, p_2, u) \equiv p_1 x_1^H(p_1, p_2, u) + p_2 x_2^H(p_1, p_2, u)$$

is

$$e(p_1, p_2, u) = p_1 u + 2\sqrt{u}\sqrt{p_1}\sqrt{p_2} \tag{5 − 6}$$

The expenditure function can also be obtained from $(5-2)$, using the fact that $v(p_1, p_2, e(p_1, p_2, u)) = u$: in this case, that means that equation $(5-2)$ implies that

$$u = \frac{(\sqrt{p_2 + e(p_1, p_2, u)} - \sqrt{p_2})^2}{p_1} \tag{5 − 7}$$

which can be solved for $e(p_1, p_2, u)$; doing so yields equation $(5-6)$.

Conversely, the indirect utility function can be obtained from equation $(5-6)$ and the fact that $e(p_1, p_2, v(p_1, p_2, y)) = y$, which here becomes

$$y = p_1 v(p_1, p_2, y) + 2\sqrt{v(p_1, p_2, y)}\sqrt{p_1}\sqrt{p_2} \tag{5 − 8}$$

which can be solved for $v(p_1, p_2, y)$ to yield equation $(5-2)$.