GS/ECON 5010 section "B"' Answers to Assignment 4 November 2014

$Q1$. Suppose that the aggregate demand curve by men for some good has the equation

$$Q^M = \frac{100}{p^2}$$

where Q^M is the aggregate quantity demanded by men, and p the price they pay. The aggregate demand curve of women, for the same product, is

$$Q^W = \frac{100}{p^3}$$

where Q^W is the aggregate quantity demanded by women.

A monopoly is able to supply the good at a constant marginal cost of $MC = 1$ (in unlimited quantities). Compare the price paid by men, and the price paid by women in the following two situations :

(i) The monopoly can charge different prices to men and women (who are not able to resell the good).

(ii) The monopoly must charge the same price to all buyers.

$A1$. The "elasticity mark–up" rule is the easiest way to solve this question : a single–price monopoly should set its price equal to

$$\frac{\epsilon}{\epsilon - 1} MC$$

where ϵ is the absolute value of the own–price elasticity of demand, and MC is the marginal cost.

Both men and women have constant–elasticity demand curves, with the own–price elasticity of demand being 2 for men and 3 for women. Since the marginal cost equals 1, the monopoly should charge men a price of

$$p^M = \frac{\epsilon^M}{\epsilon^M - 1} = 2 \tag{1-1}$$

and women a price of

$$p^W = \frac{\epsilon^W}{\epsilon^W - 1} = 1.5 \tag{1-2}$$

in case (i), when it can charge separate prices to men and women.

In this case, it will sell $\frac{100}{2^2} = 25$ units to men, and $\frac{100}{(1.5)^3} = 29.63$ units to women, for total profit of $\pi^{PD} = 25(2-1) + 29.63(1.5-1) = 39.815$.

In case (ii), the monopoly faces a single demand curve, of the form

$$Q = \frac{100}{p^2} + \frac{100}{p^3} \tag{1-3}$$

Differentiating

$$\frac{\partial Q}{\partial p} = -\frac{200}{p^3} - \frac{300}{p^4} \tag{1-4}$$

so that its own–price elasticity of demand

$$\epsilon \equiv -\frac{\partial Q}{\partial p}\frac{p}{Q} = \frac{2p+3}{p+1} \tag{1-5}$$

and therefore

$$\epsilon - 1 = \frac{p+2}{p+1} \tag{1-6}$$

so that the optimum mark–up rule becomes

$$p = \frac{\epsilon}{\epsilon - 1} = \frac{2p+3}{p+2} \tag{1-7}$$

which can be solved for p as

$$p = \sqrt{3} \approx 1.732 \tag{1-8}$$

In this case, the monopoly's total quantity sold is

$$Q = \frac{100}{(\sqrt{p})^2} + \frac{100}{(\sqrt{p})^3} \approx 52.578 \tag{1-9}$$

with total profits of

$$(p-1)Q \approx 0.732 * 52.578 = 38.49 \tag{1-10}$$

(As must be the case), the ability to price discriminate (case (i)) leads to higher profits for the monopoly than being required to sell all units at the same price (case (ii)).

$Q2$. In a duopoly, suppose that each firm has the same production technology : if they pay a fixed cost of $F > 0$, they can produce as much output as they wish, at a marginal cost of zero. (So the total cost of producing $q > 0$ units is a constant F, whereas the cost of producing nothing is zero.)

If the market demand curve has the equation

$$Q^d = B - p$$

what are the equilibria if the firms behave as Cournot duopolists, choosing quantities simultaneously and non–cooperatively?

$A2$. If both firms produce positive quantities of output, then the fact that the demand curve is linear, and that the marginal cost is constant (at 0) means that equation $(4-15)$ of Jehle and Reny can be plugged in : each firm would choose to produce the same quantity q of output, for which

$$q = \frac{B}{3} \tag{2-1}$$

In this case, the equilibrium price would be

$$p = B - 2q = \frac{B}{3} \tag{2-2}$$

so that each firm would make profits of

$$\pi = pq - F = \frac{B^2}{9} - F \tag{2-3}$$

But firms will only choose to produce positive levels of output if they earn positive profits : they always have the option of producing nothing,and avoiding the fixed cost F. Therefore, the output levels defined by equation $(2-1)$ constitute a Cournot equilibrium only if the profits defined by equation $(2-3)$ are non–negative, that is if

$$F \leq \frac{B^2}{9} \tag{2-4}$$

What would happen if equation $(2-4)$ did not hold? In this case, there **cannot** be an equilibrium in which both firms produced positive levels of output. But there might be an equilibrium in which only one firm produced a positive level of output. If firm #1 knows that it is the only firm selling a positive quantities, it will act like a single–price monopoly, and choose an output level

$$q^m = \frac{B}{2} \tag{2-5}$$

In this case, the market price would be

$$p^m = B - q^m = \frac{B}{2} \tag{2-6}$$

and firm 1 would earn non–negative profits if and only if

$$\pi^m = p^m q^m - F = \frac{B^2}{4} - F \geq 0 \tag{2-7}$$

So when

$$\frac{B^2}{9} < F \leq \frac{B^2}{4} \tag{2-8}$$

the Cournot outcome is for one firm to choose the monopoly output q^m, and for the other firm to choose to produce nothing.

If F is so large that it exceeds $B^2/4$, then no firm will ever choose to produce anything.

But there is one more detail to check. If firm 1 were to produce the monopoly output q^m, are we sure that firm 2 would choose not to produce?

If firm 2 did produce a positive quantity of output, then its best quantity to produce is its best reaction to $q_1 = q^m$,

$$q_2 = \frac{B}{2} - \frac{q^m}{2} = \frac{B}{4} \tag{2-9}$$

In this case, total output would be

$$Q = q^m + q_2 = \frac{3B}{4} \tag{2-10}$$

so that

$$p = \frac{B}{4} \tag{2-11}$$

and firm 2 would earn profits of

$$\pi_2 = pq_2 - F = \frac{B^2}{16} - F \tag{2-12}$$

So as long as

$$F \geq \frac{B^2}{16} \tag{2-13}$$

firm 1 will be assured of keeping its monopoly position : if firm 1 chooses an output of q^m, firm 2 will choose not to produce anything.

But condition $(2-13)$ is not exactly the opposite of condition $(2-4)$. It is possible that both $(2-4)$ **and** $(2-13)$ hold. And in that situation there will be **multiple** Cournot equilibria : it is an equilibrium for firms to choose $q_1 = q_2 = \frac{B}{3}$, but it is also an equilibrium for firms to choose $q_1 = \frac{B}{2}; q_2 = 0$ or $q_1 = 0; q_2 = \frac{B}{2}$.

In summary, then, the nature of the Cournot equilibrium depends on how large are the fixed costs :

(i) if $F < \frac{B^2}{16}$ then there is a unique Cournot equilibrium in which $q_1 = q_2 = \frac{B}{3}$

(ii) if $\frac{B^2}{16} \leq F \leq \frac{B^2}{9}$ there are multiple Cournot equilibria : $q_1 = q_2 = \frac{B}{3}$ and $q_1 = \frac{B}{2}; q_2 = 0$ and $q_1 = 0; q_2 = \frac{B}{2}$

(iii) if $\frac{B^2}{9} < F < \frac{B^2}{4}$ then the only Cournot equilibria involve one firm choosing the monopoly output and the other firm producing nothing

(iv) if $F > \frac{B^2}{4}$ then the only Cournot equilibrium is for neither firm to choose to produce anything

$Q3$. Another model of duopoly is that of **von Stackelberg**, in which firms choose output levels **sequentially**. That is, firm 1 chooses its output **first**, and cannot change its output after it has made its choice. Firm 2 then observes what output level firm 1 has chosen, and then chooses its own output level. What output levels would the 2 firms choose, if they behaved in this manner, if the demand and technology were as in question #2 above, with $F = 10$ and $B = 12$?

$A3$. In this case, $B^2/16 = 9 < F$, so that the answer to question #2 above can be applied immediately.

If firm #1 were to choose the monopoly level of output, $q^m = \frac{B}{2} = 6$, then firm #2 will not choose to produce. Firm 2's best response to $q_1 = 6$ is

$$q_2 = \frac{B}{2} - \frac{q_1}{2} = 3 \tag{3-1}$$

which would lead to a market price of

$$p = 12 - 6 - 3 = 3 \tag{3-2}$$

which means firm 2's profit would be

$$\pi_2 = pq_2 - F = (3)(3) - 10 = -1 < 0 \tag{3-3}$$

So firm 2 would choose not to produce, if firm 1 had committed initially to produce the single–price monopoly level of output $q^m = 6$.

That must be the equilibrium. Firm 1 cannot do better than it would as a monopoly. And choosing the monopoly level of output, in this case, leads to the follower (firm #2) choosing not to enter, which means that the leader (firm #1) can do just as well as if it were a monopoly.

$Q4$. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of A units of good 1 and B units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = (x_1^1)^3 (x_2^1)^6$$

$$u^2(x_1^2, x_2^2) = (x_1^2)^4 (x_2^2)^2$$

where x_j^i is person i's consumption of good j?

$A4$. Both people here have Cobb–Douglas preferences. The two people's marginal rates of substitution can be written

$$MRS_{xy}^1 = \frac{u_1^1}{u_2^1} = \frac{x_2^1}{2x_1^1} \tag{4 – 1}$$

$$MRS_{xy}^2 = \frac{u_1^2}{u_2^2} = \frac{2x_2^2}{x_1^2} \tag{4 – 2}$$

Efficiency in this exchange economy requires that the two people's marginal rates of substitution be equal :

$$\frac{x_2^1}{2x_1^1} = \frac{2x_2^2}{x_1^2} \tag{4 – 3}$$

Since $x_1^2 = A - x_1^1$ and $x_2^2 = B - x_2^1$, equation $(4-3)$ can be written

$$\frac{x_2^1}{2x_1^1} = \frac{2(B - x_2^1)}{A - x_1^1} \tag{4 – 4}$$

or

$$x_1^2 = \frac{4Bx_1^1}{4A + 3x_1^1} \tag{4 – 5}$$

That's it. Equation $(4-5)$ defines an upward–sloping curve in the Edgeworth box, which goes through the corners of the box, and which stays above the diagonal of the box.

$Q5$. What are the allocations in the core of the following 3–person, 2–good economy?

Person 1 regards the two goods as **perfect substitutes**.

Person 2 and person 3 regard the two goods as **perfect complements**.

The endowments of the three people are $\mathbf{e}^1 = (3, 0), \mathbf{e}^2 = (3, 0), \mathbf{e}^3 = (0, 6)$.

$A5$. Any allocation in the core must be Pareto efficient. Since person 2 and person 3 regard both goods as perfect complements, efficiency requires that $x_1^i = x_2^i$ for $i = 2, 3$. The total quantity available of each good is 6. So person 1 gets a consumption bundle $\mathbf{x}^1 = (6 - x_1^2 - x_1^3, 6 - x_2^2 - x_2^3)$: the requirements that $x_1^2 = x_2^2$ and $x_1^3 = x_2^3$ therefore imply that $x_1^1 = x_2^1$.

So the facts that (i) person 2 and 3 regard the two goods as perfect complements ; (ii) person 1 does not regard the 2 goods as perfect complements ; (iii) the aggregate endowment of each good is the same : together imply that any Pareto optimal allocation must be of the form

$$\mathbf{x}^1 = (a, a) \quad ; \quad \mathbf{x}^2 = (b, b) \quad ; \quad \mathbf{x}^3 = (c, c) \tag{5-1}$$

where

$$a + b + c = 6 \tag{5-2}$$

Any allocation in the core must be individually rational : it must offer each person at least as high a utility level as she could get from consuming her original endowment. Since person 1's preferences can be represented by the utility function $u^1(\mathbf{x}^1) = x_1^1 + x_2^1$, her allocation must give a utility level which is at least as high as $e_1^1 + e_2^1 = 3$. So it must be the case that

$$a \geq 1.5 \tag{5-3}$$

otherwise person 1 would prefer to consume her endowment bundle $(3, 0)$.

How high can a be? If person 2 and 3 formed a coalition to block an allocation $\{(a, a), (b, b), (c, c)\}$, the coalition would have to give person 3 at least c units of each good : otherwise she would be better off not joining the coalition. If person 3 gets (c, c) from a 2–person coalition with person 2, that leaves the rest of the coalition's total endowment $\mathbf{e}^2 + \mathbf{e}^3 = (3, 6)$ to person 2. So person 2 would get the allocation $(3 - c, 6 - c)$ if he joined the blocking coalition. He'd be willing to do this if and only if he preferred that allocation to the original allocation $\{(a, a), (b, b), (c, c)\}$ which the coalition is trying to block. That is, he'd join if and only if

$$3 - c > b$$

That means that the original allocation must have $3 - c \leq b$ if it is in the core : otherwise a coalition of people #2 and #3 would block it. So $3 - c \leq b$ is required for an allocation to be in the core, or (since $a = 6 - (b + c)$)

$$a \leq 3 \tag{5-4}$$

How high can b be? If b were too high, people #1 and #3 could form a coalition to block the original allocation $\{(a, a), (b, b), (c, c)\}$. The blocking coalition would have to offer (c, c) to person

#3 to get her to be willing to join. That leaves $(3-c, 6-c)$ for person #1. So person #1 will be willing to join the coalition, and to block the original allocation, if she prefers this new consumption bundle : that is, if

$$(3-c)+(6-c) > 2b$$

If the allocation $\{(a,a),(b,b),(c,c)\}$ is in the core, it cannot be blocked by a coalition of #1 and #3, so it must be the case that $(3-c)+(6-c) \leq 2a$, which is equivalent to $9-2c \leq 2a$, meaning $a+c \geq 4.5$. Since $a+b+c=6$, therefore another necessary condition for an allocation $\{(a,a),(b,b),(c,c)\}$ to be in the core is

$$b \leq 1.5 \tag{5-5}$$

Finally, what could a coalition of people #1 and #2 do? Not much. Since this coalition has none of good 2 in its aggregate endowment, it can block an allocation only if $b=0$: person #2 can't do better than $(0,0)$ if she can't get any of good #2. So the only allocations which a coalition of person #1 and person #2 can block are allocations in which $b=0$ and in which $a<3$. (The latter condition ensures that person #1 would prefer forming a coalition with person #2, and getting the coalition's entire endowment $(6,0)$, to the consumption bundle (a,a) she gets in the original allocation.)

Since all possible blocking coalitions have been considered, the core consists of all allocations of the form $\{(a,a),(b,b),(c,c)\}$, for which $a \geq 0, b \geq 0, c \geq 0$ and for which

$$a+b+c=6 \quad ; \quad 1.5 \leq a \leq 3 \quad ; \quad 0 < b \leq 1.5 \tag{5-6}$$