GS/ECON 5010 section "B"' Answers to Assignment 1 October 2015

*Q*1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

There are three different goods : potatoes, rice and noodles. (So a consumption bundle (x_1, x_2, x_3) is a bundle with x_1 kilograms of potatoes, x_2 kilograms of rice and x_3 kilograms of noodles.)

Each kilogram of potatoes has 1000 calories, each kilogram of rice has 800 calories and each kilogram of noodles also has 800 calories.

If bundle $\mathbf{x}$ has strictly more calories than bundle $\mathbf{y}$, then the person prefers $\mathbf{x}$ strictly to the bundle $\mathbf{y}$.

If the bundles $\mathbf{x}$ and $\mathbf{y}$ have the same number of calories, then the person prefers strictly whichever bundle contains strictly more rice.

If two bundles have the same number of calories, and the same quantity of rice, then the person is indifferent between them.

*A*1. To check strict monotonicity, we need to check whether the following two properties hold for the preferences described : (i) if there are two bundles $\mathbf{x}$ and $\mathbf{z}$ with $\mathbf{x} \geq \mathbf{z}$, then the consumer must rank $\mathbf{x}$ as at least as good as $\mathbf{z}$; (ii) if $\mathbf{x} >> \mathbf{z}$, then the consumer must prefer strictly $\mathbf{x}$ to $\mathbf{z}$.

Both properties hold here. If $\mathbf{x} \geq \mathbf{z}$, then $1000x_1 + 800x_2 + 800x_3 \geq 1000z_1 + 800z_2 + 800z_3$, so that bundle $\mathbf{x}$ has at least as many calories as bundle $\mathbf{z}$. As well, if $\mathbf{x} \geq \mathbf{z}$, then $x_2 \geq z_2$, so that bundle $\mathbf{x}$ has at least as much rice as bundle $\mathbf{z}$.

And if $\mathbf{x} >> \mathbf{z}$, then $1000x_1 + 800x_2 + 800x_3 > 1000z_1 + 800z_2 + 800z_3$, so that bundle $\mathbf{x}$ has strictly more calories than bundle $\mathbf{z}$, and must be preferred strictly by the consumer.

To check convexity, suppose that $\mathbf{x}$ is at least as good as $\mathbf{z}$. Preferences are convex if (and only if) **any** bundle $t\mathbf{x} + (1-t)\mathbf{z}$ is ranked at least as good as $\mathbf{z}$, for some scalar t with $0 \leq t \leq 1$.

Now if $\mathbf{x}$ is at least as good as $\mathbf{z}$, then bundle $\mathbf{x}$ has at least as many calories as bundle $\mathbf{z}$, and at least as much rice as in bundle $\mathbf{z}$. The number of calories in any bundle $t\mathbf{x} + (1-t)\mathbf{z}$ must be somewhere in between the number of calories in bundle $\mathbf{x}$ and the number of calories in bundle $\mathbf{z}$. [The number of calories in bundle $t\mathbf{x} + (1-t)\mathbf{z}$ is a weighted average of the number of calories in bundles $\mathbf{x}$ and $\mathbf{z}$, with weights of t on $\mathbf{x}$ and $(1-t)$ on $\mathbf{z}$.] Also, the quantity of rice in bundle $t\mathbf{x} + (1-t)\mathbf{z}$ must be somewhere in between the quantity of rice in bundle $\mathbf{x}$ and the quantity of rice in bundle $\mathbf{z}$. So if $\mathbf{x}$ is ranked at least as good as bundle $\mathbf{z}$, then bundle $t\mathbf{x} + (1-t)\mathbf{z}$ must have at least as many calories as bundle $\mathbf{z}$, and at least as much rice as is in bundle $\mathbf{z}$, and so must be ranked as at least as good as bundle $\mathbf{z}$. That shows that preferences here **are** convex.

$Q2$. Are the preferences represented by the utility function below strictly monotonic? Convex? Explain briefly.

$$U(x_1, x_2) = \min(x_1, x_2 - \frac{1}{x_1})$$

$A2$. To check strict monotonicity, note that the function $x_2 - \frac{1}{x_1}$ is strictly increasing in x_1 and in x_2. [Take the partial derivatives to see this.]

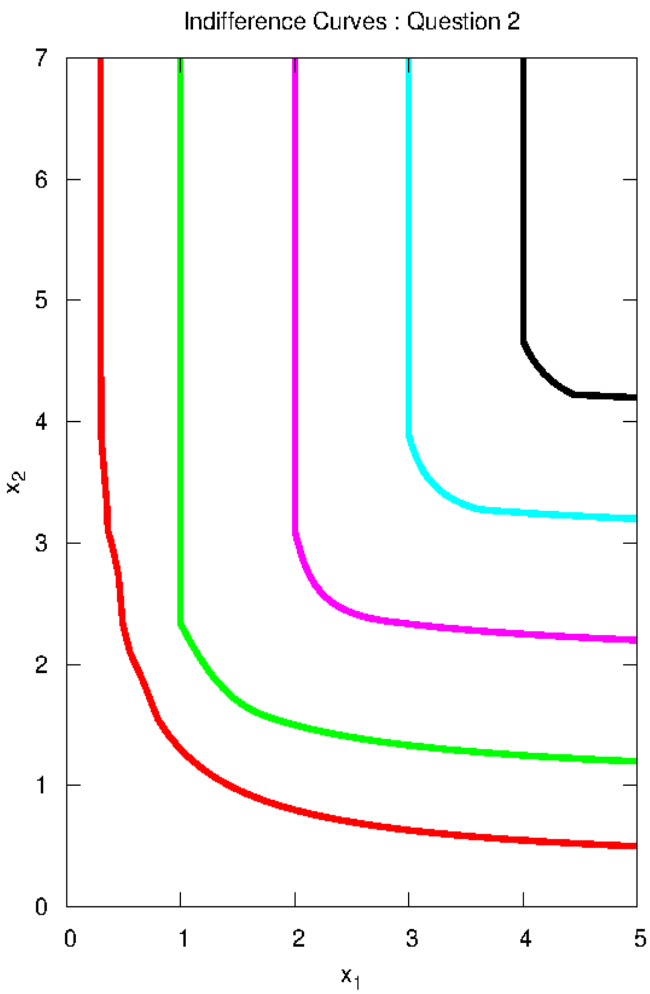
So if $\mathbf{x} \geq \mathbf{z}$ then $x_1 \geq z_1$ and $x_2 - \frac{1}{x_1} \geq z_2 - \frac{1}{z_1}$ so that $U(x_1, x_2) \geq U(z_1, z_2)$. Similarly, if $\mathbf{x} >> \mathbf{z}$, then $x_1 > z_1$ and $x_2 - \frac{1}{x_1} > z_2 - \frac{1}{z_1}$ so that $U(x_1, x_2) > U(z_1, z_2)$. The preferences are strictly monotonic.

One way to check convexity is to look at the indifference curves for this utility function. The indifference curve for a utility level of u here has two pieces : a vertical line at $x_1 = u$, at and above the point $(u, u + \frac{1}{u})$, and a line with the equation $x_2 = u + \frac{1}{x_1}$ at and below the point $(u, u + \frac{1}{u})$.

[Why the point $(u, u + \frac{1}{u})$ for the point where the pieces connect? The two pieces connect at a point at which $x_1 = x_2 - \frac{1}{x_1} = u$, and solving the equation $x_2 - \frac{1}{u} = u$ yields $x_2 = u + \frac{1}{u}$.]

That indifference curve gets less steep as we move down and to the right : it is vertical above the kink at $(u, u + \frac{1}{u})$, and to the right of the kink, the absolute value of slope of the curve $x_2 = u + \frac{1}{x_1}$ is $|\frac{dx_2}{dx_1}| = \frac{1}{(x_1)^2}$, which gets smaller as x_1 increases. So the preferences are convex (but actually not strictly convex, since the vertical segment of the indifference curve is a straight line).

[The figure below shows a few of the indifference curves for these preferences.]



some indifference curves for $U(x_1, x_2) = \min\left(x_1, x_2 - \frac{1}{x_1}\right)$

$Q3$. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \frac{x_1 x_2}{x_1 + x_2} + \log x_3$$

$A3$. For this person, the first–order conditions for utility maximization are

$$u_1 = [\frac{x_2}{x_1 + x_2}]^2 = \lambda p_1 \tag{3 – 1}$$

$$u_2 = [\frac{x_1}{x_1 + x_2}]^2 = \lambda p_2 \tag{3 – 2}$$

$$u_3 = \frac{1}{x_3} = \lambda p_3 \tag{3 – 3}$$

Equations $(3-1)$ and $(3-2)$ imply that

$$x_1 = [\frac{\sqrt{p_2}}{\sqrt{p_1}}]x_2 \tag{3 – 4}$$

Substituting for x_1 into equation $(3-1)$ from equation $(3-4)$,

$$[\frac{\sqrt{p_1}}{\sqrt{p_1} + \sqrt{p_2}}]^2 = \lambda p_1 \tag{3 – 5}$$

or

$$\lambda = [\sqrt{p_1} + \sqrt{p_2}]^{-2} \tag{3 – 6}$$

Now use $(3-6)$ to substitute for λ into equation $(3-3)$:

$$x_3 = [\sqrt{p_1} + \sqrt{p_2}]^2 p_3^{-1} \tag{3 – 7}$$

Equation $(3-7)$ is the Marshallian demand function for good 3 : it expresses quantity demanded of good 3 as a function of the prices of the three goods, and of income. Actually, income does not appear in equation $(3-7)$: the income elasticity of Marshallian demand for good 3 is 0 in this case.

To get the Marshallian demand functions for the other two goods, use the budget constraint that

$$p_1 x_1 + p_2 x_2 + p_3 x_3 = y \tag{3 – 8}$$

or

$$p_1 x_1 + p_2 x_2 = y - p_3 x_3 = y - [\sqrt{p_1} + \sqrt{p_2}]^2 \tag{3 – 9}$$

From equation $(3-4)$,

$$p_1 x_1 + p_2 x_2 = \sqrt{p_1 p_2} x_2 + p_2 x_2 \tag{3 – 10}$$

so that equation $(3-9)$ can be written

$$x_2 = \frac{y - [\sqrt{p_1} + \sqrt{p_2}]^2}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \tag{3 – 11}$$

or

$$x_2 = \frac{y}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} - \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_2}} \tag{3 – 12}$$

which is the Marshallian demand function for good 2. Substitution from equation $(3-4)$ into $(3-12)$ yields the Marshallian demand function for good 1,

$$x_1 = \frac{y}{\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} - \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}} \tag{3 – 13}$$

[Notice that the system of three Marshallian demand functions defined by equations $(3-13)$, $(3-12)$ and $(3-7)$ satisfy the requisite properties : each of the functions is homogeneous of degree zero in income and prices together, and the "adding–up" property, $p_1 x_1(\mathbf{p}, y) + p_2 x_2(\mathbf{p}, y) + p_3 x_3(\mathbf{p}, y) = y$ holds.]

*Q*4. Calculate a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2) = min(x_1, 2x_2)$$

*A*4. Calculus will not help much with this question : here the two goods are **perfect complements**, with L–shaped indifference curves. The consumer's optimum — the "tangency" of an indifference curve with the budget line — will always require her to choose a consumption bundle which is at the kink an indifference curve.

That is, if $x_1 > 2x_2$, buying any more of good 1 costs her money, but makes her no better off. If $2x_2 > x_1$, then buying more of good 2 makes her no better off and costs her money. So she will **always** choose a consumption bundle which is at the kink on an indifference curve, that is a bundle (x_1, x_2) for which

$$x_1 = 2x_2 \qquad (4-1)$$

She also wants to be on her budget line : since these preferences are strictly monotonic, she wants to spend all the budget available. That means that her consumption bundle must satisfy the budget constraint

$$p_1x_1 + p_2x_2 = y \qquad (4-2)$$

with equality.

So to find her Marshallian demand functions, solve equations $(4-1)$ and $(4-2)$ for x_1 and x_2. For example, substitution for x_2 from equation $(4-1)$ into equation $(4-2)$ yields

$$p_1x_1 + p_2\frac{x_1}{2} = y \qquad (4-3)$$

or

$$x_1^M(p_1, p_2, y) = \frac{2y}{2p_1 + p_2} \qquad (4-4)$$

and

$$x_2^M(p_1, p_2, y) = \frac{y}{2p_1 + p_2} \qquad (4-5)$$

$Q5$. Find the expenditure function, Hicksian demand functions and indirect utility function for the preferences of question #4 above.

$A5$. If the consumer solves the dual problem of minimizing the cost of a utility level u, again she will want to locate at the kink of the indifference curve, that is at a point where

$$x_1 = 2x_2 = u \tag{5 – 1}$$

where u is the required level of utility.

So equation $(5-1)$ yields immediately the Hicksian demand functions,

$$x_1^H(p_1, p_2, u) = u \tag{5 – 2}$$

$$x_2^H(p_1, p_2, u) = \frac{u}{2} \tag{5 – 3}$$

Then the expenditure function $e(p_1, p_2, u)$ is solved as

$$e(p_1, p_2, u) = p_1 x_1^H(p_1, p_2, u) + p_2 x_2^H(p_1, p_2, u) = (2p_1 + p_2)\frac{u}{2} \tag{5 – 4}$$

Differentiation of $(5-4)$ with respect to the prices shows that tese expenditure and Hicksian demand functions satisfy Shepherd's lemma.

The indirect utility function can be derived from the answer to question #4 above : here $v(p_1, p_2, y) = u(x_1^M(p_1, p_2, y), x_2^M(p_1, p_2, y))$. Here the utility is the minimum of $\frac{2y}{2p_1+p_2}$ and $2\frac{y}{2p_1+p_2}$, which are equal, and which equal

$$v(p_1, p_2, y) = \frac{2y}{2p_1 + p_2} \tag{5 – 5}$$

The "duality" relations between the expenditure and indirect utility functions are satisfied here : equations $(3-4)$ and $(3-5)$ satisfy

$$v(p_1, p_2, e(p_1, p_2, u)) = u \tag{5 – 6}$$

and

$$e(p_1, p_2, v(p_1, p_2, y)) = y \tag{5 – 7}$$