GS/ECON 5010 section "B"' Answers to Assignment 2 October 2015

*Q*1. Could the following three functions be Marshallian demand functions for a consumer with well–behaved preferences? Explain briefly.

$$x_1(\mathbf{p},y) = yp_1^{-1} - 2^{4/3}p_1^{-2/3}p_2^{1/3}p_3^{1/3}$$
$$x_2(\mathbf{p},y) = 2^{1/3}p_1^{1/3}p_2^{-2/3}p_3^{1/3}$$
$$x_3(\mathbf{p},y) = 2^{1/3}p_1^{1/3}p_2^{1/3}p_3^{-2/3}$$

*A*1. There are two properties that need to be checked : (*i*) Walras's Law (the value of total quantities demanded must equal the person's income, whatever are the prices and income), (*ii*) a negative semi–definite substitution matrix. [These two properties imply that the functions must be homogeneous of degree 0 in prices and income together.]

Any set of "candidates for demand functions" which satisfy these two properties represent Marshallian demand functions for some consumer with well–behaved preferences. On the other hand, if either of these properties is violated (for any values of $\mathbf{p}$ or y), then the candidates for demand functions cannot represent the Marshallian demand functions of a consumer with well–behaved preferences.

(*i*) For these three functions,

$$\mathbf{p}\cdot\mathbf{x}(\mathbf{p},y) = y - 2^{4/3}p_1^{1/3}p_2^{1/3}p_3^{1/3} + 2^{1/3}p_1^{1/3}p_2^{1/3}p_3^{1/3} + 2^{1/3}p_1^{1/3}p_2^{1/3}p_3^{1/3} \tag{1-1}$$

But

$$2^{1/3} + 2^{1/3} = 2[2^{1/3}] = 2^{4/3}$$

so that equation $(1-1)$ implies that $\mathbf{p}\cdot\mathbf{x}(\mathbf{p},y) = y$, whatever are the prices $\mathbf{p}$, or the income available y. So Walras's Law must be satisfied by this system of functions.

(*ii*) To check the substitution matrix, we need to find the 3–by–3 matrix S, in which element S_{ij} equals

$$\frac{\partial x_i}{\partial p_j} + x_j(\mathbf{p},y)\frac{\partial x_i}{\partial y}$$

For these functions,

$$S = \begin{pmatrix} -\frac{2}{3}2^{1/3}p_1^{-5/3}p_2^{1/3}p_3^{1/3} & \frac{1}{3}2^{1/3}p_1^{-2/3}p_2^{-2/3}p_3^{1/3} & \frac{1}{3}2^{1/3}p_1^{-2/3}p_2^{1/3}p_3^{-2/3} \\ \frac{1}{3}2^{1/3}p_1^{-2/3}p_2^{-2/3}p_3^{1/3} & -\frac{2}{3}2^{1/3}p_1^{1/3}p_2^{-5/3}p_3^{1/3} & \frac{1}{3}2^{\frac{1}{3}}p_1^{1/3}p_2^{-2/3}p_3^{-2/3} \\ \frac{1}{3}2^{1/3}p_1^{-2/3}p_2^{1/3}p_3^{-2/3} & \frac{1}{3}2^{1/3}p_1^{1/3}p_2^{-2/3}p_3^{-2/3} & -\frac{2}{3}2^{1/3}p_1^{1/3}p_2^{1/3}p_3^{-5/3} \end{pmatrix}$$

or

$$S = \frac{1}{3}2^{\frac{1}{3}}p_1^{-5/3}p_2^{-5/3}p_3^{-5/3}\begin{pmatrix} -2(p_2)^2(p_3)^2 & p_1p_2(p_3)^2 & p_1(p_2)^2p_3 \\ p_1p_2(p_3)^2 & -2(p_1)^2(p_3)^2 & (p_1)^2p_2p_3 \\ p_1(p_2)^2p_3 & (p_1)^2p_2p_3 & -2(p_1)^2(p_2)^2 \end{pmatrix} \tag{1-2}$$

This matrix is symmetric. The elements on the diagonal are all negative. The determinant of the whole matrix is 0. And the determinant of the 2–by–2 sub–matrix in the upper left–hand corner is

$$D_2 = 2^{\frac{1}{3}} p_1^{1/3} p_2^{1/3} p_3^{7/3} > 0$$

So the substitution matrix is symmetric, and it is negative semi–definite.

The three functions possess all the properties required for a system of Marshallian demand functions, so that they could represent the Marshallian demand functions of a consumer with well–behaved preferences. [In fact, in this case they are the Marshallian demand functions for a consumer with utility function $u(\mathbf{x}) = x_1 - \frac{1}{x_2 x_3}$.]

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times. (For example, the third row indicates that the consumer chose the bundle $\mathbf{x} = (30, 30, 10)$ when the price vector was $\mathbf{p} = (10, 5, 10)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	10	10	5	20	20	20
2	5	10	10	30	15	25
3	10	5	10	30	30	10
4	5	5	10	30	20	20

*A*2. One way of finding the violations of the strong and weak axioms of revealed preference is first to construct the matrix, in which the element M_{ij} is the cost of bundle $\mathbf{x}^j$ at prices $\mathbf{p}^i$. Here that matrix is

$$\begin{pmatrix} 500 & 575 & 650 & 600 \\ 500 & 550 & 550 & 550 \\ 500 & 625 & 550 & 600 \\ 400 & 475 & 400 & 450 \end{pmatrix}$$

Using this matrix, the bundle $\mathbf{x}^i$ is directly revealed preferred to the bundle $\mathbf{x}^j$ if $M_{ii} \geq M_{ij}$. For example, row 3 of the matrix has $X_{33} > X_{31}$: that means that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^1$, since bundle $\mathbf{x}^1$ was affordable in period 3 (it cost \$500), and the person instead chose bundle $\mathbf{x}^3$.

The first row shows that bundle $\mathbf{x}^1$ is not directly revealed preferred to any of the other bundles, since all of the other three bundles are outside the period–1 budget line with equation $10x_1 + 10x_2 + 5x_3 = 500$.

The second row shows that bundle $\mathbf{x}^2$ is directly revealed preferred to all 3 other bundles, since each of the other bundles is affordable at period–2 prices $(5, 10, 10)$, with period–2 income \$550.

The third row shows that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle 1, but not to the other two bundles.

And the fourth row shows that bundle $\mathbf{x}^4$ is directly revealed preferred to bundles 1 and 3.

There are **no** violations here, either of *WARP*, or of *SARP*. The bundles here can actually be ranked. For this consumer, bundle $\mathbf{x}^2$ is on a higher indifference curve than bundle $\mathbf{x}^4$, which is on a higher indifference curve than $\mathbf{x}^3$ which is on a higher indifference curve than $\mathbf{x}^1$.

*Q*3. If a person has a constant coefficient of relative risk aversion equal to β, what is the probability of winning π which must be offered the person to make her just willing to accept the following bet? The bet : with probability π the person's initial wealth increases by a factor of 4 (from W_0 to $4W_0$) but with probability $1-\pi$ she loses all her initial wealth.

*A*3. Since this person has a constant coefficient of relative risk aversion, the solution to this problem will not depend on the exact level of her initial wealth.

If her initial wealth is W_0, then if she accepts the bet she will have wealth of $4W_0$ with probability π and wealth of 0 with probability $1-\pi$, giving her an expected utility of

$$EU = \frac{1}{1-\beta}[\pi(4W_0)^{1-\beta} + (1-\pi)0^{1-\beta}] \tag{3-1}$$

If she does not take the bet, her expected utility will be

$$U_0 = \frac{1}{1-\beta}[W_0^{1-\beta}] \tag{3-2}$$

If $\beta < 1$, then $0^{1-\beta} = 0$, so that the value of π which makes these two expressions ($(3-1)$ and $(3-2)$) equal is the solution to

$$\pi 4^{1-\beta} = 1 \tag{3-3}$$

or

$$\pi = 4^{\beta-1} \tag{3-4}$$

Note that this expression $(3-4)$ makes sense only if $\beta < 1$: if the person is risk averse enough that $\beta \geq 1$, then the possibility of losing all her wealth drives her expected utility in the "bad" state to $-\infty$, so that she would be unwilling to accept the bet no matter how likely is the "good" outcome.

*Q*4. If a person has a constant coefficient of relative risk aversion equal to 2, what would be the highest amount that she would be willing to pay to insure completely against loss of half of her wealth, if she perceived the probability of that loss as equalling some π (with $0 < \pi < 1$)?

*A*4. The person's von–Neumann–Morgenstern utility–of–wealth function is

$$U(W) = -W^{-1}$$

if she has a constant coefficient of relative risk aversion of 2. Her alternatives are to purchase the complete insurance against the loss at some total price P, leaving her with the loss (of $W_0/2$) fully covered in the "bad" state, and with wealth of $4W_0 - P$ in either state of the world, giving her expected utility of

$$EU_I = -(W_0 - P)^{-1} \qquad (4-1)$$

(if her initial wealth is W_0), or doing without any insurance, giving her an expected utility of

$$EU_N = -(1-\pi)(W_0)^{-1} - \pi(\frac{W_0}{2})^{-1} \qquad (4-2)$$

If she is just willing to purchase the insurance, she should be indifferent between these alternatives. Setting expression $(4-1)$ equal to expression $(4-2)$, the maximum price P which she is willing to pay satisfies the equation

$$\frac{1}{W_0 - P} = \frac{1-\pi}{W_0} + \frac{2\pi}{W_0} \qquad (4-3)$$

which implies that

$$P = \frac{\pi}{1+\pi} W_0 \qquad (4-4)$$

Notice, as expected, that the price she is willing to pay is proportional to her wealth (since she has a CRR von Neumann–Morgenstern utility–of–wealth function), and that the price she is willing to pay for insurance exceeds the expected loss, which is $\frac{\pi W_0}{2}$.

*Q*5. A von Neumann–Morgenstern expected utility maximizer has a utility–of–wealth function

$$U(W) = -\frac{1}{W+A}$$

where A is some positive constant. What is the certainty equivalent for her of a gamble which doubles her wealth with probability π, and leaves her with zero wealth with probability $1-\pi$?

*A*5. The certainty equivalent to the gamble, CE is the certain level of wealth which gives her the same expected utility as the gamble. The expected utility of the gamble is

$$EU = -\frac{\pi}{2W+A} - \frac{1-\pi}{A} \qquad (5-1)$$

Her expected utility from having CE dollars for sure is

$$U(CE) = -\frac{1}{CE+A} \qquad (5-2)$$

So if $U(CE) = EU$, then equations $(5-1)$ and $(5-2)$ imply that

$$CE = 2\pi W \frac{A}{2(1-\pi)W + A} \qquad (5-3)$$

[For this person, the coefficient of relative risk aversion is

$$R_R = \frac{W}{W+A} \qquad (5-4)$$

The larger is A, the **lower** is her coefficient of relative risk aversion. Equation $(5-3)$ is consistent with this result : the higher is A, the **higher** is the certainty equivalent to the gamble (and the **lower** is her risk premium).

Also, as her wealth W increases, equation $(5-4)$ shows that her coefficient of relative risk aversion increases. Consistent with that notion, equation $(5-3)$ shows that, CE/W decreases with W, so that her risk premium for the gamble goes up more than proportionally with her wealth.]