GS/ECON 5010 Section "B" Answers to Assignment 3 F2015

*Q*1. Is the production function

$$f(\mathbf{x}) = 1 - \frac{1}{(x_1+1)(x_2+1)(x_3+1)}$$

weakly separable? Strongly separable? Explain briefly.

*A*1. In this example, the marginal products of the three inputs are

$$MP_1 = [A(x_1+1)]^{-1} \qquad (1-1)$$

$$MP_2 = [A(x_2+1)]^{-1} \qquad (1-2)$$

$$MP_3 = [A(x_3+1)]^{-1} \qquad (1-3)$$

where

$$A \equiv (x_1+1)(x_2+1)(x_3+1) \qquad (1-4)$$

Therefore the marginal rate of technical substitution between any two inputs i and j is

$$MRTS_{ij} = \frac{(x_j+1)}{(x_i+1)} \qquad (1-5)$$

which is a function only of x_i and x_j. So the production function is strongly (and weakly) separable.

*Q*2. Derive the cost function for the production function

$$f(\mathbf{x}) = 1 - \frac{1}{(x_1+1)(x_2+1)(x_3+1)}$$

*A*2. The first–order condition for cost minimization is that

$$MRTS_{ij} = \frac{w_i}{w_j} \qquad (2-1)$$

for any pair of inputs i and j. This condition, and equations $(1-1)-(1-4)$ imply that

$$1 + x_2 = \frac{w_1}{w_2}(1+x_1) \qquad (2-2)$$

$$1 + x_3 = \frac{w_1}{w_3}(1+x_1) \qquad (2-3)$$

when the firm minimizes its costs. Substitution from $(2-2)$ and $(2-3)$ into the definition of the production function yields

$$y = 1 - \frac{1}{(1+x_1)^3\frac{(w_1)^2}{w_2 w_3}} \qquad (2-4)$$

which can be re–arranged into

$$x_1 = (\frac{w_2 w_3}{w_1^2})^{1/3}(1-y)^{-1/3} - 1 \qquad (2-5)$$

which is the conditional input demand for input #1. Substituting for x_1 from $(2-5)$ into $(2-2)$ and $(2-3)$ yields

$$x_2(\mathbf{w}, y) = (\frac{w_1 w_3}{w_2^2})^{1/3}(1-y)^{-1/3} - 1 \qquad (2-6)$$

$$x_3(\mathbf{w}, y) = (\frac{w_1 w_2}{w_3^2})^{1/3}(1-y)^{-1/3} - 1 \qquad (2-7)$$

as the conditional demand functions for inputs # 2 and 3. Since

$$C(\mathbf{w}, y) = w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y) + w_3 x_3(\mathbf{w}, y)$$

equations $(2-5)$ – $(2-7)$ imply that here

$$C(\mathbf{w}, y) = 3(w_1 w_2 w_3)^{1/3}(1-y)^{-1/3} - w_1 - w_2 - w_3 \qquad (2-8)$$

[You can check that equations $(2-5)$ – $(2-8)$ satisfy Shephard's Lemma.]

[Expression $(2-8)$ holds only if there is an interior solution to the firm's cost minimization problem.

But corner solutions are possible : if the conditional input demand for input i, as defined in equations $(2-5)$ – $(2-7)$, is negative, then we have a corner solution. For example, if $w_2 w_3 < (1-y)([w_1]^2)$, then the x_1 defined in equation $(2-5)$ is negative. In such a case — input 1 is relatively expensive, and the required output level y is relatively low — cost minimization would lead to a corner solution, in which $x_1 = 0$.

In fact, if y is small enough, one of the x_i's defined by equations $(2-5)$–$(2-7)$ **must** be negative[1].

However, there are 6 possible corner solutions which may arise[2], so that I will avoid details here (and I hope you avoided them as well).]

*Q*3. Derive the profit function for the production function

$$f(\mathbf{x}) = 1 - \frac{1}{(x_1+1)(x_2+1)(x_3+1)}$$

*A*3. The easiest way to find the profit function here is to use the solution to question #2 above. A competitive firm should choose its output level y to maximize

$$py - C(\mathbf{w}, y)$$

[1] except in the single case in which $w_1 = w_2 = w_3$

[2] $x_1 = 0, x_2 > 0, x_3 > 0$; $x_1 = 0, x_2 = 0, x_3 > 0$; $x_1 = 0, x_2 > 0, x_3 = 0$; $x_1 > 0, x_2 = 0, x_3 > 0$; $x_1 > 0, x_2 = 0, x_3 = 0$; $x_1 > 0, x_2 > 0, x_3 = 0$

with respect to y. Maximization of

$$py - 3(w_1 w_2 w_3)^{1/3}(1-y)^{-1/3} - w_1 - w_2 - w_3$$

has a first–order condition

$$p - (w_1 w_2 w_3)^{-1/3}(1-y)^{-4/3} \quad (3-1)$$

(which is the condition that output be chosen so that price equals marginal cost). Solving equation $(3-1)$ for y results in

$$y(\mathbf{w}, p) = 1 - p^{-3/4}(w_1 w_2 w_3)^{1/4} \quad (3-2)$$

which is the firm's supply function.

The firm's profit function is

$$\pi(\mathbf{w}, p) = py(\mathbf{w}, p) - C(\mathbf{w}, y(\mathbf{w}, p)) \quad (3-3)$$

or (substituting from $(3-2)$ into $(3-3)$, and using expression $(2-8)$ from the answer to the previous question)

$$\pi(\mathbf{w}, p) = p - 4p^{1/4}(w_1 w_2 w_3)^{1/4} + w_1 + w_2 + w_3 \quad (3-4)$$

*Q*4. What is the long–run industry supply curve in an industry in which there are t firms of type t, where $t = 1, 2, 3, \ldots 1000$, in which the total cost function of a type–t firm is

$$TC^t(y) = t\frac{y^2}{2} + y$$

where y is the firm's output?

*A*4. A type–t firm has a marginal cost function of

$$MC^t(y) = ty + 1 \quad (4-1)$$

and an average cost function of

$$AC^t(y) = \frac{ty}{2} + 1 \quad (4-2)$$

so that marginal cost is increasing in y and always greater than average cost (except at $y = 0$). So a firm's supply curve is its marginal cost curve. Solving the first–order condition $MC = p$ means that

$$ty + 1 = p$$

or

$$y^t(p) = \frac{p-1}{t} \quad (4-3)$$

So all firms (regardless of their type t) will be willing to supply positive quantities of output if (and only if) $p > 1$. The aggregate quantity supplied by all type–t firms is

$$ty^t(p) = p - 1 \qquad t = 1, 2, 3, \ldots, 1000 \tag{4-4}$$

which means that the aggregate industry supply, which equals

$$\sum_{t=1}^{1000} ty^t(p)$$

is

$$Y(p) = 1000(p - 1) \qquad p \geq 1 \tag{4-5}$$

and aggregate supply is 0 if $p < 1$.

*Q*5. What membership fee F, and what unit price p, should a monopoly charge in the following market?

There are equal numbers of two types (indexed by t) of consumer. The monopoly must charge the same membership fee to all customers, and must charge the same unit price to all customers. The monopoly produces its product at zero cost (and so tries to maximize its total revenue). Each customer gets a utility of

$$U = u_0$$

if she buys nothing from the monopoly, and

$$U = u_0 - F - px + tx - \frac{1}{2}x^2$$

if she pays a membership fee F and buys x units of the monopoly's product at a price of p each.

For half the people, $t = 10$, and for the other half $t = 20$.

To purchase the monopoly's product (at a price of p per unit), a customer must pay the membership fee F. [So that a customer's two options are to pay the fee F, and then choose how many units x of the monopoly's product to buy at a price p, or (*ii*) not to pay the membership fee, and not to buy anything from the monopoly.]

*A*5. Working backwards, how much would a customer buy of the monopoly's product, if she had chosen to pay the membership fee F? She would choose x so as to maximize

$$U = u_0 - F - px + tx - \frac{1}{2}x^2$$

which means that

$$x = t - p \tag{5-1}$$

resulting in a level of utility of

$$U^m = u_0 - F - p(t - p) + t(t - p) - \frac{1}{2}(t - p)^2 = u_0 - F + \frac{1}{2}(t - p)^2 \tag{5-2}$$

So the customer will choose to purchase a membership if (and only if) U^m is higher than the utility u_0 she would get if she did not buy a membership (and was unable to buy any of the monopoly's product). So she will choose to buy a membership if and only if

$$(t-p)^2 \geq 2F \tag{5 − 3}$$

Now if both types of people strictly preferred membership (so that $(5-3)$ held as a strict inequality for both $t = 10$ and $t = 20$), then the monopoly could raise F a little, and make more money per customer without losing any customers. (Notice that the quasi–linear utility function means that sales per customer, defined in equation $(5-1)$ do not vary with the membership fee F.)

So the monopoly should always set the fee as high as it can. That leaves the monopoly with 2 choices : set $F = (10-p)^2/2$ and have all the people choose to buy memberships, or to set $F = (20-p)^2/2$ and have only the high demand types ($t = 20$) choose to buy membership.

In the first case, the monopoly's total revenue is proportional to

$$R = 2F + p(10-p) + p(20-p) \tag{5 − 4}$$

where the first term is membership revenue, the second is sales revenue from type–10 customers, and the third is sales revenue from type–20 customers. Since it chooses $F = (10-p)^2/2$, its total revenue is

$$(10-p)^2 + p(10-p) + p(20-p) \tag{5 − 5}$$

Choosing p to maximize expression $(5-5)$ means setting $p = 5$, yielding revenue of 125.

In the second case, its total revenue is proportional to

$$F + p(20-p)$$

which equals

$$\frac{(20-p)^2}{2} + p(20-p) \tag{5 − 6}$$

when it chooses the highest membership fee possible, $F = (20-p)^2/2$. Maximizing expression $(5-6)$ with respect to p means giving the product away free to members : $p = 0$. This policy results in a membership fee of 200 — and revenue of 200.

The firm's best policy here is to give away its product to members, but to charge a very high membership fee (of 200), so as to attract only the demanders with the strongest taste for the firm's product and to extract all those customers' consumer surplus.