GS/ECON 5010 section "B"' Answers to Assignment 4 November 2015

$Q1$. Another model of duopoly is that of **von Stackelberg**, in which firms choose output levels **sequentially**. That is, firm 1 chooses its output quantity **first**, and cannot change that quantity after it has made its choice. Next, firm 2 observes what quantity firm 1 has chosen, and then chooses its own output quantity. What quantities would the 2 firms choose, if they behaved in this manner, if the cost of production (for each firm) were 0, and if the aggregate demand for the firms' homogeneous product were

$$Q = 12 - p$$

(where p is the market price, and Q the aggregate quantity demanded)?

$A1$. The key here is that firm 1, if it is clever, will anticipate firm 2's reaction to its own behavior. That is, firm 2's profit maximizing choice of output q_2 will be on its reaction function $q_2^R(q_1)$ to firm 1's initial quantity choice. That means that firm 1 knows that firm 2's quantity will be $q_2^R(q_1)$, and total industry output will be $q_1 + q_2^R(q_1)$, when it makes its initial decision on what quantity q_1 to produce.

Working backwards, firm 2 will choose q_2 so as to maximize its profits

$$\pi_2 = (12 - q_1 - q_2)q_2 \qquad (1-1)$$

treating q_1 as given. Setting the derivative of $(1-1)$ with respect to q_2 equal to 0,

$$(12 - q_1) - 2q_2 = 0 \qquad (1-2)$$

or

$$q_2^R(q_1) = 6 - \frac{q_1}{2} \qquad (1-3)$$

[Note that firm 2's reaction here is just the reaction function for a Cournot oligopolist, $q_2^R(q_1) = \frac{a-c}{2b} - \frac{q_1}{2}$ when $a = 12$, $b = 1$ and $c = 0$.]

Firm 1 should anticipate that firm 2's choice will be defined by equation $(1-3)$, so that firm 1's own profit will be

$$\pi_1 = (12 - q_1 - q_2^R(q_1))q_1 \qquad (1-4)$$

Firm 1 should choose its output level so as to maximize expression $(1-4)$ with respect to q_1, recognizing that q_2 will depend on its own choice of q_1.

Since $\frac{\partial q_2^R}{\partial q_1} = -1/2$ (from equation $(1-3)$), the the derivative of $(1-4)$ with respect to q_1 is

$$(12 - q_1 - [6 - \frac{q_1}{2}]) - (1 - \frac{1}{2})q_1$$

or

$$6 - q_1$$

So firm 1's profit is maximized if it chooses a quantity level of 6, with firm 2 reacting to that choice by choosing its own quantity level of

$$q_2 = 6 - \frac{q_1}{2} = 3$$

The market price will be 3, and the firms' profits will be $\pi_1 = 18$ and $\pi_2 = 9$.

*Q*2. Solve for the equilibrium quantities in a **3**–firm Stackelberg model, with the demand and cost functions from question #1 above.

[That is, firm 1 commits first to its quantity q_1. Firm 2 observes q_1 , and then commits to its own quantity q_2. Finally firm 3 observes q_1 and q_2, and then chooses its profit–maximizing output quantity q_3.]

*A*2. As in the previous question, the equilibrium here can be derived by working backwards, starting with firm 3's reaction to the quantities q_1 and q_2 chosen by the other firms. Firm 3 will try to maximize its profit

$$\pi_3 = (12 - q_1 - q_2 - q_3)q_3 \tag{2 – 1}$$

with respect to q_3 treating q_1 and q_2 as constants. Taking the derivative of $(2 - 1)$ with respect to q_3, and setting it equal to 0, yields

$$q_3^R(q_1, q_2) = 6 - \frac{q_1 + q_2}{2} \tag{2 – 2}$$

which again is the reaction function of a Cournot oligopolist, $q_3^R(q_1, q_2) = \frac{a-c}{2b} - \frac{q_1+q_2}{2}$ when $a = 12$, $b = 1$, $c = 0$.

Firm 2 should anticipate that firm 3 will choose its quantity according to equation $(2 - 2)$, which means that firm 2's own profits will be

$$\pi_2 = (12 - q_1 - q_2 - [6 - \frac{q_1 + q_2}{2}])q_2 \tag{2 – 3}$$

The derivative of $(2 - 3)$ with respect to q_2 is

$$6 - \frac{q_1}{2} - q_2 \tag{2 – 4}$$

so that firm 2's optimal choice of quantity (when it takes firm 1's quantity choice as given, and anticipates firm 3's choice) is

$$q_2^R(q_1) = 6 - \frac{q_1}{2} \tag{2 – 5}$$

Now firm 1, when it makes its quantity decision, realizes that firm 2's subsequent quantity decision will be determined by $(2 - 5)$, and then firm 3's decision by $(2 - 2)$. So firm 1's profits, when it chooses a quantity of q_1, will be

$$\pi_1 = (12 - q_1 - [6 - \frac{q_1}{2}] - [6 - \frac{q_1}{2} - \frac{6 - q_1/2}{2}])q_1 \tag{2 – 6}$$

or

$$\pi_1 = (3 - \frac{q_1}{4})q_1 \qquad (2-7)$$

Taking the derivative of $(2-7)$ with respect to q_1, and setting it equal to zero, yields

$$q_1 = 6$$

which means (from equation $(2-5)$) that

$$q_2 = 3$$

and (from equation $(2-2)$

$$q_3 = 1.5$$

so that $p = 12 - 6 - 3 - 1.5 = 1.5$ and $\pi_1 = 9$, $\pi_2 = 4.5$, $\pi_3 = 2.25$.

*Q*3. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of A units of good 1 and B units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = 1 - \frac{1}{x_1^1} - \frac{1}{x_2^1}$$

$$u^2(x_1^2, x_2^2) = \log(x_1^2) + x_2^2$$

where x_j^i is person i's consumption of good j?

*A*3. Inside the Edgeworth box, the contract curve consists of the points for which the two people's marginal rates of substitution are equal. Given these utility functions

$$\frac{\partial u^1}{\partial x_1^1} = \frac{1}{(x_1^1)^2} \qquad (3-1)$$

$$\frac{\partial u^1}{\partial x_2^1} = \frac{1}{(x_1^2)^2} \qquad (3-2)$$

so that

$$MRS^1 = [\frac{x_2^1}{x_1^1}]^2 \qquad (3-1)$$

and

$$\frac{\partial u^2}{\partial x_1^2} = \frac{1}{x_1^2} \qquad (3-4)$$

$$\frac{\partial u^2}{\partial x_2^2} = 1 \qquad (3-5)$$

so that

$$MRS^2 = \frac{1}{x_1^2} \qquad (3-6)$$

Everything that person 1 does not consume gets allocated to person 2, so that

$$x_1^2 = A - x_1^1 \tag{3-7}$$

which means that the efficiency condition $MRS^1 = MRS^2$ can be written

$$\frac{[x_1^1]^2}{[x_2^1]^2} = A - x_1^1 \tag{3-8}$$

or

$$x_2^1 = \frac{x_1^1}{\sqrt{(A - x_1^1)}} \tag{3-9}$$

Equation (3–9) defines an upward–sloping curve, staring at the bottom left corner of the Edgeworth box : equation $(3-9)$ implies that $x_2^1 = 0$ when $x_1^1 = 0$.

But the curve defined by equation $(3-9)$ does **not** go through the top right corner of the box : as $x_1^1 \to A$, the value of x_2^1 defined by equation $(3-9)$ approaches infinity.

So the here the contract curve hits the top edge of the Edgeworth Box (at the point at which $\frac{x_1^1}{\sqrt{(A-x_1^1)}} = B$), and then moves along the top of the box. Because person 2's MRS approaches ∞ as $x_1^2 \to 0$ (no matter how small is x_2^2), there will always be efficient allocations in which person 2 consumes positive quantities of good 1, but no good 2.

*Q*4. What are the allocations in the core of the following 3–person, 2–good economy?

Person i's preferences can be represented by the utility function $u^i(x_1^i, x_2^i)$, where

$$u^1(x_1^1, x_2^1) = x_1^1$$

$$u^2(x_1^2, x_2^2) = x_1^2 x_2^2$$

$$u^3(x_1^3, x_2^3) = x_2^3$$

and the endowment vectors of the three people are $\mathbf{e}^1 = (0,4), \mathbf{e}^2 = (4,0), \mathbf{e}^3 = (2,2)$.

*A*4. Since person 1 does not like good 2, and person 3 does not like good 1, the only Pareto optimal allocations are those in which person #1 gets a consumption bundle $(a, 0)$, person 2 gets a consumption bundle $(6-a, 6-c)$, and person 3 gets a consumption bundle $(0, c)$ — with $0 \leq a \leq 6$, and $0 \leq c \leq 6$.

And any allocation in the core must be Pareto optimal.

Core allocations must be individually rational : they must offer each person at least as high a level of utility as she would get from consuming her endowment vector e^i. Since person #1 and person #2 get the same utility from their endowment bundles as from the bundle $(0,0)$, the only constraint imposed by individual rationality is that person #3 get utility at least as high as she would get from her endowment bundle $(2,2)$. So individual rationality imposes the constraint that $c \geq 2$ on the efficient allocations defined in the first paragraph.

If person 1 and person 3 form a coalition, the coalition would give all of its units of good #1 to person 1, and all of its units of good #2 to person 3. Therefore, a coalition of person 1 and person 3 would give $(2,0)$ to person #1 and $(0,6)$ to person #3.

So an allocation $\{(a,0),(6-a,6-c),(0,c)\}$ will be blocked by a coalition of person 1 and person 3 if : $a < 2$, or $a = 2$ and $c < 6$. [Note that it must be true that $c \leq 6$ in any allocation : there are only 6 units available of each good.]

On the other hand, if $a > 2$, then the allocation cannot be blocked by a coalition of person 1 and person 3 : such a coalition has only 2 units of good #1 to give to person 1.

What allocations could person 1 and person 2 block together? The coalition has a total endowment of $(4,4)$, which it would divide between person #1 and person #2 if it were seeking to block some allocation. If person 1 were offered a units of good #1 in some allocation, she would have to get at least a units of good 1 from any coalition seeking to block that allocation. So if she joined a coalition with person #2, and got a units of good #1 from that potential blocking coalition, then person #2 would get the rest of the coalition's endowment : $(4-a,4)$. So person 2 would be willing to join the coalition if (and only if) it gave him more utility than the proposed allocation. That means the coalition of person #1 and person #2 could block the allocation if and only if

$$(4-a)4 > (6-a)(6-c) \qquad (4-1)$$

The left side of inequality $(4-1)$ is what person #2 gets from joining the blocking coalition, and the right side is what he gets from the original proposed allocation. Therefore, if an allocation is in the core – which means it can't be blocked by any coalition — inequality $(4-1)$ **can't** hold. Inequality $(4-1)$ not holding is the same thing as inequality $(4-2)$ being true :

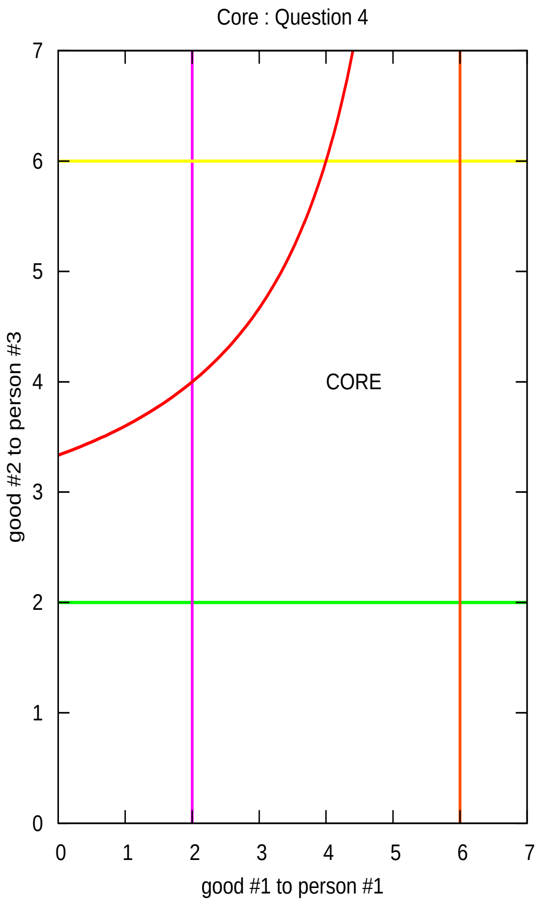
$$2a - ac + 6c \leq 20 \qquad (4-2)$$

What kinds of allocation might be blocked by a coalition of person #2 and person #3? That coalition would have a total endowment of only 2 units of good 2. So the best bundle it could offer to person #3 would be the bundle $(0,2)$. But that's a bundle she could get from her own endowment. It was already argued above the individual rationality imposes the constraint $c \geq 2$ on any allocations in the core. And any allocation in which $c \geq 2$ cannot be blocked by a coalition of person #2 and person #3.

That's all the blocking coalitions that there are. So the set of allocations in the core are all those allocations $\{(a,0),(6-a,6-c),(0,c)\}$ for which : $a \geq 2$, and $c \geq 2$, and $2a - ac + 6c \leq 20$.

Note that the core contains some allocations which leave person #2 rather badly off : the allocation $\{(5.9,0),(0.1,0.1),(0,5.9)\}$ is in the core, for example.

The accompanying figure illustrates the core in $(a-c)$ space. So in the figure, the variable on the horizontal axis is how much of good 1 is allocated to person #1, and the variable on the vertical axis is how much of good 2 is allocated to person #3. The core (in the figure) is the set of all allocations which are : above the green horizontal line, below the yellow horizontal line, to the the right of the purple vertical line, to the left of the orange vertical line **and** below (and to the right of) the upward-sloping red curve. (This last curve, the upward-sloping one, defines a and c such that the allocation $(a, 0), (6 - a, 6 - c), (0, c)$ is just on the verge of being blocked by a coalition of person #1 and person #2.)


Core : Question 4
CORE
good #2 to person #3
good #1 to person #1
0
1
2
3
4
5
6
7

*Q*5. Find a competitive equilibrium to a 2–good, 3–million–person economy, in which 1 million people have preferences and endowments like person 1 in the previous question (# 4), 1 million people have preferences and endowments like person 2 in the previous question, and 1 million people have preferences and endowments like person 3 in the previous question. [That is, find a competitive equilibrium to an economy which is the economy of question #4 cloned one million times.]

*A*5. A competitive equilibrium price vector is a price vector (p_1, p_2) which equates the demand for each good with the total endowment of the good.

In this economy, the income of each type–1 person is $4p_2$, the income of each type–2 person is $4p_1$, and the income of each type–3 person is $2(p_1 + p_2)$.

Type–1 people spend all their money on good 1, the only good they like. So each type–1 person will demand $4\frac{p_2}{p_1}$ units of good 1. Type–2 people have Cobb–Douglas preferences, so that each type–2 person's quantity demanded of good 1 is $y/2p_1$, and since her income y equals $4p_1$, she will demand 2 units of good 1 (regardless of the prices). Therefore, total demand for good 1 (divided by one million) is the sum of demand by type–1 people and demand by type–2 people,

$$4\frac{p_2}{p_1} + 2$$

The total endowment is 6 (million), so that the excess demand for good 1 is

$$Z_1(p_1, p_2) = 4\frac{p_2}{p_1} + 2 - 6 \qquad (5-1)$$

In equilibrium, total excess demand must equal zero, so that equation $(5-1)$ implies that the market for good #1 clears only if

$$p_1 = p_2$$

Walras's Law implies that excess demand for good 2 must equal 0 if aggregate excess demand for good 1 is 0. So the only possible equilibrium price vectors are those for which $p_1 = p_2$.

For example, $\mathbf{p} = (1, 1)$ is an equilibrium price vector. In this situation, each person's income is 4. Person 1's consumption bundle is $(4, 0)$, person 2's is $(2, 2)$ and person 3's is $(0, 4)$.

As it must be, this allocation is in the core of the economy described in question #4.