Q1. Are the preferences described below strictly monotonic? Convex? Explain briefly.

In comparing any two bundles $\mathbf{x}$ and $\mathbf{z}$, the person strictly prefers bundle $\mathbf{x}$ to bundle $\mathbf{z}$ if $x_1 + x_2 > z_1 + z_2$, and strictly prefers bundle $\mathbf{z}$ to bundle $\mathbf{x}$ if $z_1 + z_2 > x_1 + x_2$.

If $x_1 + x_2 = z_1 + z_2$, the person strictly prefers bundle $\mathbf{x}$ to bundle $\mathbf{z}$ if $x_1 > z_1$, and strictly prefers bundle $\mathbf{z}$ to bundle $\mathbf{x}$ if $z_1 > x_1$.

A1. These preferences cannot be represented by a utility function . (They are not continuous.)

But they are strictly monotonic, and they are convex.

If $\mathbf{x} \geq \mathbf{z}$, then the person must weakly prefer $\mathbf{x}$ to $\mathbf{z}$. The only ways that she could prefer $\mathbf{z}$ strictly are if $z_1 + z_2 > x_1 + x_2$, or if $x_1 + x_2 = z_1 + z_2$ and $z_1 > x_1$. Neither of those conditions are possible if $\mathbf{x} \geq \mathbf{z}$.

If $\mathbf{x} >> \mathbf{z}$, then $x_1 + x_2 > z_1 + z_2$, so that $\mathbf{x}$ must be preferred strictly to $\mathbf{z}$.

The two previous paragraphs together imply that these preferences are strictly monotonic.

Now suppose that **both** $\mathbf{y}$ and $\mathbf{z}$ are preferred weakly to $\mathbf{x}$.

In order for that to be true, it must be the case that $y_1 + y_2 \geq x_1 + x_2$, and $z_1 + z_2 \geq x_1 + x_2$, and it must also be the case that if $z_1 + z_2 = x_1 + x_2$ then $z_1 \geq x_1$, and if $y_1 + y_2 = x_1 + x_2$ then $y_1 \geq x_1$.

So if $\mathbf{w} \equiv t\mathbf{y} + (1-t)\mathbf{z}$, then $w_1 + w_2 = t(y_1 + y_2) + (1-t)(z_1 + z_2) \geq t(x_1 + x_2) + (1-t)(x_1 + x_2) = x_1 + x_2$.

So the only way that $\mathbf{x}$ could be preferred strictly to $\mathbf{w}$ is if $w_1 + w_2 = x_1 + x_2$ and $x_1 > w_1$. But $w_1 + w_2 = x_1 + x_2$ only if $y_1 + y_2 = x_1 + x_2$ **and** $z_1 + z_2 = x_1 + x_2$. And in that case, it must be true (if $\mathbf{y}$ is preferred weakly to $\mathbf{x}$, and if $\mathbf{z}$ is preferred weakly to $\mathbf{x}$) that $y_1 \geq x_1$ and $z_1 \geq x_1$. Which means that

$$w_1 \equiv ty_1 + (1-t)x_1 \geq tx_1 + (1-t)x_1 = x_1$$

which means that $\mathbf{w}$ is preferred weakly to $\mathbf{x}$.

So if $\mathbf{y}$ and $\mathbf{z}$ are both in the "at least as good as " set for $\mathbf{x}$, then it must be true that $w \equiv t\mathbf{y} + (1-t)\mathbf{z}$ (with $0 \leq t \leq 1$) is also in the "at least as good as" set for $\mathbf{x}$. So these preferences are convex.

Q2. Are the preferences described below strictly monotonic? Convex? Explain briefly.

The person likes more of each good, but she also wants the quantities of the 2 goods to be as close as possible. In particular, her utility function can be represented as the sum $x_1 + x_2$ of the quantities of goods 1 and 2, **minus** b times the absolute value of the **difference** $|x_1 - x_2|$ between the quantities of the two goods, where $0 < b < 1$.

A2. These preferences can be represented by the utility function

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 - b(x_1 - x_2) & \text{if } x_1 > x_2 \\ x_1 + x_2 - b(x_2 - x_1) & \text{if } x_2 > x_1 \end{cases}$$

As long as $b < 1$, these preferences are strictly monotonic : when $x_1 > x_2$, then $u_1 = (1-b) > 0$ and $u_2 = (1+b) > 0$, and if $x_2 > x_1$ then $u_1 = 1 + b > 0$ and $u_2 = 1 - b > 0$.

They also are convex (although not strictly convex). The easiest way to see this is to look at the indifference curves. The slope of an indifference curve is $-u_1/u_2$ [when x_1 is graphed on the horizontal axis, and x_2 on the vertical].

So the indifference curves have a slope of $(1+b)/(1-b)$ above the 45–degree line, and a slope of $(1-b)/(1+b)$ below the 45–degree line. That means (since $1 > b > 0$) that they get less steep moving from above to below the 45–degree line.

With only 2 goods, and strictly monotonic preferences, a sufficient condition for convexity of the preferences is that the slope of the indifference curve does not increase in absolute value as we move down and to the right.

Q3. What are a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = \log x_1 + \log x_2 + \sqrt{x_3} \qquad ?$$

A3. The first–order conditions for utility maximization are

$$\frac{1}{x_1} = \lambda p_1 \tag{3-1}$$

$$\frac{1}{x_2} = \lambda p_2 \tag{3-2}$$

$$\frac{1}{2\sqrt{x_3}} = \lambda p_3 \tag{3-3}$$

From equations $(3-1)$ and $(3-2)$,

$$x_2 = \frac{p_1}{p_2} x_1 \tag{3-4}$$

Since equation $(3-1)$ implies that $\lambda = \frac{1}{p_1 x_1}$, equation $(3-3)$ can be written

$$\frac{1}{2}(x_3)^{-0.5} = \frac{p_3}{p_1 x_1} \tag{3-5}$$

or

$$x_3 = \frac{(p_1 x_1)^2}{4(p_3)^2} \qquad (3-6)$$

From equations $(3-4)$ and $(3-6)$, the consumer's budget constraint $p_1x_1 + p_2x_2 + p_3x_3 = y$ can be written

$$2p_1x_1 + \frac{1}{4}p_3[\frac{p_1x_1}{p_3}]^2 = y \qquad (3-7)$$

which is a quadratic equation in x_1. If I denote

$$z \equiv p_1x_1$$

and

$$\beta \equiv \frac{1}{4p_3}$$

then equation $(3-7)$ becomes

$$\beta z^2 + 2z - y = 0 \qquad (3-8)$$

Using the quadratic formula, the solution to equation $(3-8)$ is

$$z = \frac{1}{\beta}(\sqrt{1+\beta y} - 1) \qquad (3-9)$$

Since $x_1 = \beta/p_1$, equation $(3-9)$ implies that

$$x_1 = \frac{4p_3}{p_1}(\sqrt{1 + \frac{y}{4p_3}} - 1) \qquad (3-10)$$

which is the Marshallian demand function for good 1.

Equation $(3-4)$ then yields the Marshallian demand function for good 2,

$$x_2 = \frac{4p_3}{p_2}(\sqrt{1 + \frac{y}{4p_3}} - 1) \qquad (3-10)$$

Substitution from $(3-9)$ into $(3-6)$ gives the Marshallian demand function for good 3,

$$x_3 = 8 + \frac{y}{p_3} - 8\sqrt{1 + \frac{y}{4p_3}} \qquad (3-11)$$

Q4. What quantities of goods 1 and 2 will a person demand if her preferences can be represented by the utility function

$$u(x_1, x_2) = x_1 x_2$$

if her income is y, the price of good # 1 is \$2, and if good # 2 has the following non–linear price schedule : the first 12 units of good # 2 cost \$4 each, and each additional unit of good # 2 (above 12) costs \$1 each?

A4. In this case, the person's budget set is **not convex**. The slope of the budget line, $-p_1/p_2$, is -2 above the horizontal line $x_2 = 12$, and is $-1/2$ below this line.

This non–convexity means that the person could have **two** tangencies of her indifference curve with the (kinked) budget "line".

The trick in this question is to find which of the two tangencies is on a higher indifference curve.

First of all, suppose she chooses a consumption bundle in which she buys less than 12 units of good 2. In this case, she has Cobb–Douglas preferences, with the prices being $p_1 = 2$ and $p_2 = 4$. Then

$$x_1^M(p_1, p_2, y) = \frac{y}{2p_1} = \frac{y}{4} \qquad (4-1)$$

$$x_2^M(p_1, p_2, y) = \frac{y}{2p_2} = \frac{y}{8} \qquad (4-2)$$

yielding her a total utility of

$$v(p_1, p_2, y) = \frac{y^2}{32} \qquad (4-3)$$

[Note that equation $(4-2)$ makes sense only if $y/8 < 12$, since she has to pay the high price of 4 for good 2 only on the first 12 units purchased. So there is a tangency to this (lower, less steep) part of the budget line only if $y < 96$.]

If she chooses a consumption bundle in which $x_2 > 12$, then her budget constraint becomes

$$2x_1 + (x_2 - 12) + 48 = y \qquad (4-4)$$

since she spends \$48 on her first 12 units of good 2, and \$1 on every subsequent unit. Equation $(4-4)$ can be written

$$x_2 = y - 2x_1 - 36 \qquad (4-5)$$

which means that her utility is

$$u = x_1 x_2 = x_1(y - 2x_1 - 36) \qquad (4-6)$$

Maximizing $(4-6)$ with respect to x_1 yields the first–order condition

$$y - 36 = 4x_1 \qquad (4-7)$$

or

$$x_1 = \frac{y}{4} - 9 = \frac{y-36}{4} \tag{4-8}$$

and (from equation $(4-5)$)

$$x_2 = \frac{y}{2} - 18 = \frac{y-36}{2} \tag{4-9}$$

[Notice that equations $(4-8)$ and $(4-9)$ are the Marshallian demand functions for this consumer, if she faces prices of $p_1 = 2$ and $p_2 = 1$, if her income were, not y, but $y - 36$. In other words, she faces a price of 1 for good 2, but since she had to spend $4 - 1 = 3$ dollars extra for each of the first 12 units, her income has effectively decreased by \$36.]

[Equation $(4-9)$ makes sense only if $x_2 > 12$, which occurs only if her income is more than \$60.]

From equations $(4-8)$ and $(4-9)$, her total utility, if she has an income greater than 60, and if she chooses to buy more than 12 units of good 2, is

$$v(p_1, p_2, y) = \frac{(y-36)^2}{8} \tag{4-10}$$

Whenever her income is between 60 and 96, there are two points at which one of her indifference curves is tangent to the border of her budget set. To see which one is on a higher indifference curve, check whether expression $(4-3)$ or $(4-10)$ has a higher value.

$\frac{(y-36)^2}{8} > \frac{y^2}{32}$ if and only if $4(y-36)^2 > y^2$ if and only if $2(y-36) > y$, which is equivalent to $y = 72$.

So the person's optimal behavior is to buy $\frac{y}{8} < 12$ units of good 2 if her income is below 72, and to buy $\frac{y-36}{2} > 12$ units if her income exceeds 72.

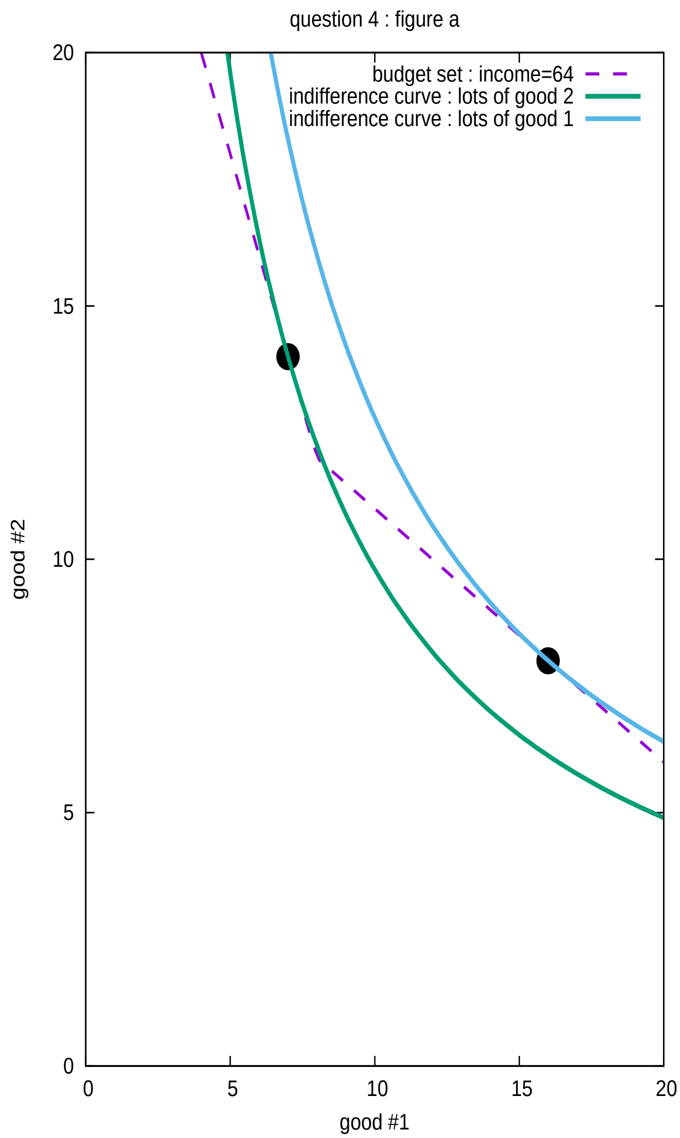
If her income were exactly 72, then she would have two equally–good choices : $x_2 = 9$ (and $x_1 = 18$), or $x_2 = 18$ (and $x_1 = 9$).

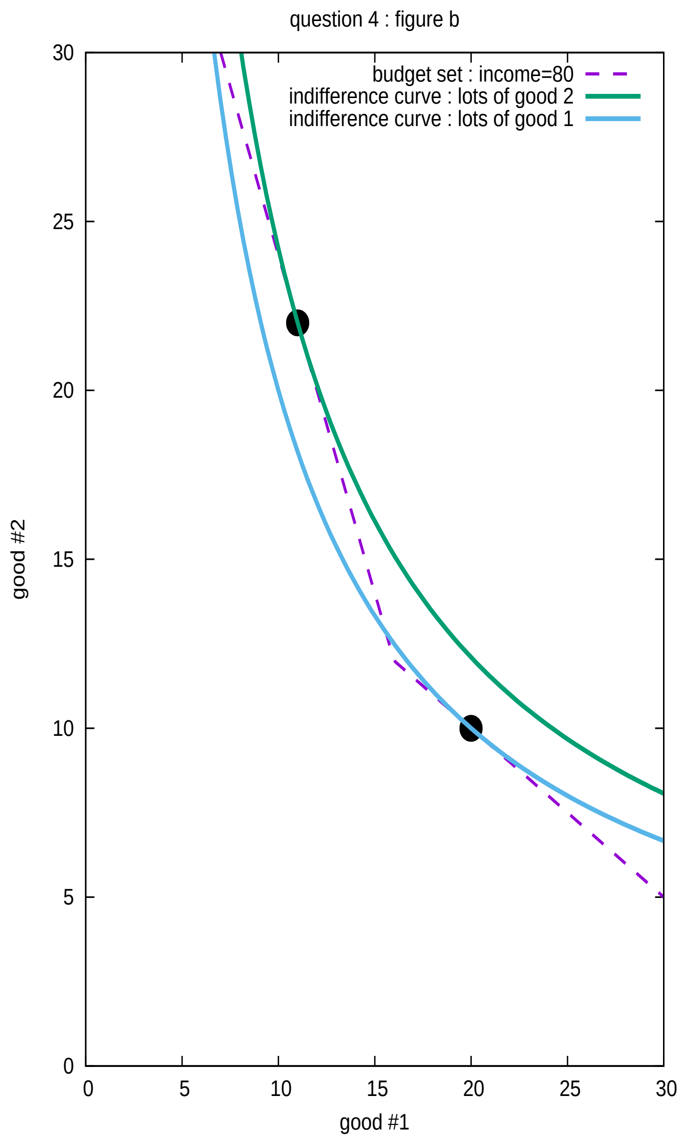
Notice that her consumption of good 2, as a function of her income y, jumps discontinuously at the income level of 72. When she is rich enough to take advantage of the volume discount for good 2, her consumption jumps from 9 to 18. (And, as a consequence, her consumption of good 1 falls discontinuously.)

Figure a below shows her budget set when her income is 64 : here there are 2 tangencies of her indifference curve with the boundary of the budget set, and she prefers the tangency in which $x_2 < 12$.

In figure b her income is 80 : again there are 2 tangencies, but here she prefers the tangency at which $x_2 > 12$.

If her income were exactly 72, then the same indifference curve would be tangent at two points on the boundary of the budget set ($x_1 = 18, x_2 = 9$ and $x_1 = 9, x_2 = 18$).


question 4 : figure a
budget set : income=64
indifference curve : lots of good 2
indifference curve : lots of good 1
good #2
good #1
0
5
10
15
20
0
5
10
15
20


question 4 : figure b
budget set : income=80
indifference curve : lots of good 2
indifference curve : lots of good 1
good #2
good #1
0
5
10
15
20
25
30
0
5
10
15
20
25
30

Q5. Derive the indirect utility function, expenditure function, and Hicksian demand function for the preferences

$$u(x_1, x_2) = \min\left[x_1(x_2)^2, (x_1)^2 x_2\right]$$

A5. Note than

$$x_1(x_2)^2 > (x_1)^2 x_2$$

if and only if

$$x_2 > x_1$$

So if $x_2 < x_1$, then $u(x_1, x_2) = x_1(x_2)^2$, and if $x_1 < x_2$ then $u(x_1, x_2) = (x_1)^2 x_2$. (If $x_1 = x_2$, then the two expressions are equal.)

With these preferences, there is a kink in the person's indifference curve, on the 45–degree line. The person's marginal rate of substitution is

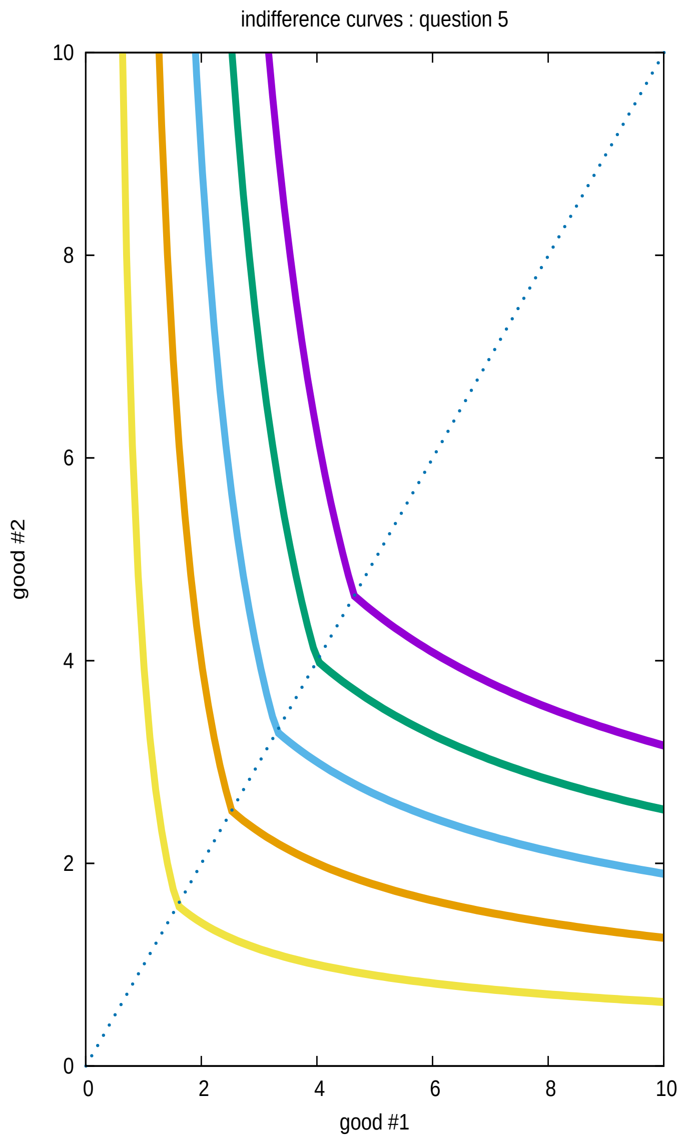
$$MRS \equiv u_1/u_2 = \frac{(x_2)^2}{2x_1x_2} = \frac{x_2}{2x_1} = \frac{1}{2}$$

at $x_1 = x_2$, if we go slightly down and to the right (where $x_2 < x_1$), and

$$MRS \equiv u_1/u_2 = \frac{2x_1x_2}{(x_1)^2} = \frac{2x_2}{x_1} = 2$$

if we go slightly up and to the left (where $x_2 > x_1$).

The figure on the next page shows some of this person's indifference curves.


indifference curves : question 5
good #2
good #1
0
2
4
6
8
10
0
2
4
6
8
10

So if the price ratio p_1/p_2 is between 0.5 and 2, then there is no tangency between the budget line and the person's indifference curve : above the 45–degree line $MRS > 2$, and below the 45–degree line $MRS < 0.5$.

That is, whenever the price ratio p_1/p_2 is between 0.5 and 2, then the budget line is "tangent" at the kink in the person's indifference curve, at the point on the budget line at which $x_1 = x_2$.

With these preferences, then, the person's behaviour has three different segments :

(a) When $p_1 > 2p_2$, the person acts as if her utility function is $u(x_1, x_2) = (x_1)^2 x_2$, which is a Cobb–Douglas utility function, resulting in Marshallian demand functions of

$$x_1^M(p_1, p_2) = \frac{2y}{3p_1} \tag{5 − 1}$$

$$x_2^M(p_1, p_2) = \frac{y}{3p_2} \tag{5 − 2}$$

(b) When $2p_2 > p_1 > p_2/2$, the person chooses the consumption bundle at the kink in her indifference curves, where $x_1 = x_2$. She acts as if she regards the two goods as perfect complements, with Marshallian demand functions

$$x_1^M(p_1, p_2) = \frac{y}{p_1 + p_2} \tag{5 − 3}$$

$$x_2^M(p_1, p_2) = \frac{y}{p_1 + p_2} \tag{5 − 4}$$

(c) When $p_2 > 2p_1$, then the person again acts as if she had Cobb–Douglas preferences, this time with $u(x_1, x_2) = x_1(x_2)^2$, yielding Marshallian demand functions

$$x_1^M(p_1, p_2) = \frac{y}{3p_1} \tag{5 − 5}$$

$$x_2^M(p_1, p_2) = \frac{2y}{3p_2} \tag{5 − 6}$$

Her indirect utility function in each of these three regions is $u(x_1^M(p_1, p_2, y), x_2^M(p_1, p_2, y))$:

(a) When $p_1 > 2p_2$, $v(p_1, p_2, y) = [x_1^M(p_1, p_2, y)]^2 x_2(p_1, p_2, y)$, which (from equations $(5 - 1)$ and $(5 - 2)$ is

$$v(p_1, p_2, y) = \frac{4}{27} y^3 (p_1)^{-2} p_2^{-1} \tag{5 − 7}$$

(b) When $2p_2 > p_1 > p_2/2$, so that $x_1 = x_2 \equiv x$, then both Cobb–Douglas functions yield the same value for utility, x^3. So (from equations $(5 - 3)$ and $(5 - 4)$)

$$v(p_1, p_2, y) = y^3 (p_1 + p_2)^{-3} \tag{5 − 8}$$

[You can check that expressions $(5 - 7)$ and $(5 - 8)$ have exactly the same value if (and only if) $p_1 = 2p_2$, so that the indirect utility function is continuous at $p_1 = 2p_2$.]

(c) When $p_2 > 2p_1$, $v(p_1, p_2, y) = x_1^M(p_1, p_2, y)][x_2(p_1, p_2, y)]^2$ which (from equations $(5-5)$ and $(5-6)$ is

$$v(p_1, p_2, y) = \frac{4}{27} y^3 (p_1)^{-1} p_2^{-2} \qquad (5-9)$$

The expenditure function can be derived, in each case, from the fact that $v(p_1, p_2, e(p_1, p_2, u)) = u$, so that

(a) When $p_1 > 2p_2$, $v(p_1, p_2, e(p_1, p_2, u)) = \frac{4}{27}[e(p_1, p_2, u)]^3 (p_1)^{-2} p_2^{-1}$, implying that

$$e(p_1, p_2, u) = (3)(4^{-1/3}) u^{1/3} (p_1)^{2/3} (p_2)^{1/3} \qquad (5-10)$$

(b) When $2p_2 > p_1 > p_2/2$, then $v(p_1, p_2, e(p_1, p_2, u)) = [e(p_1, p_2, u)]^3 (p_1 + p_2)^{-3}$, so that

$$e(p_1, p_2, u) = u^{1/3}(p_1 + p_2) \qquad (5-11)$$

(c) When $p_2 > 2p_1$, then $v(p_1, p_2, e(p_1, p_2, u)) = \frac{4}{27}[e(p_1, p_2, u)]^3 (p_1)^{-1} p_2^{-2}$ so that

$$e(p_1, p_2, u) = (3)(4^{-1/3}) u^{1/3} (p_1)^{1/3} (p_2)^{2/3} \qquad (5-12)$$

Since the Hicksian demand functions are the derivatives of the expenditure function with respect to the goods' prices, therefore

(a) When $p_1 > 2p_2$,

$$x_1^H(p_1, p_2, u) = 2(4^{-1/3}) u^{1/3} (p_1)^{-1/3} (p_2)^{1/3} = 2^{1/3} u^{1/3} (p_1)^{-1/3} (p_2)^{1/3} \qquad (5-13)$$

$$x_2^H(p_1, p_2, u) = (4^{-1/3}) u^{1/3} (p_1)^{2/3} (p_2)^{-2/3} \qquad (5-14)$$

(b) When $2p_2 > p_1 > p_2/2$, then

$$x_1^H(p_1, p_2, u) = u^{1/3} \qquad (5-15)$$

$$x_2^H(p_1, p_2, u) = u^{1/3} \qquad (5-16)$$

(c) When $p_2 > 2p_1$,

$$x_1^H(p_1, p_2, u) = (4^{-1/3}) u^{1/3} (p_1)^{-2/3} (p_2)^{2/3} \qquad (5-17)$$

$$x_2^H(p_1, p_2, u) = 2(4^{-1/3}) u^{1/3} (p_1)^{1/3} (p_2)^{-1/3} = 2^{1/3} u^{1/3} (p_1)^{1/3} (p_2)^{-1/3} \qquad (5-13)$$