Q1. Could the following three functions be Marshallian demand functions for a consumer with well–behaved preferences? Explain briefly.

$$x_1(\mathbf{p}, y) = \frac{y}{p_1} - \sqrt{\frac{p_2}{p_1}} - \sqrt{\frac{p_3}{p_1}}$$

$$x_2(\mathbf{p}, y) = \sqrt{\frac{p_1}{p_2}}$$

$$x_3(\mathbf{p}, y) = \sqrt{\frac{p_1}{p_3}}$$

A2. From the "Integrability Theorem" (theorem 2.6 in *Jehle and Reny*), these three functions represent a system of Marshallian demand functions if and only if they obey "budget balance", and the substitution matrix corresponding to these functions is symmetric and negative semi-definite.

To check budget balance,

$$p_1x_1(\mathbf{p}, y) + p_2x_2(\mathbf{p}, y) + p_3x_3(\mathbf{p}, y) = y_1 - \sqrt{p_1p_2} - \sqrt{p_1p_3} + \sqrt{p_1p_2} + \sqrt{p_1p_3} = y \qquad (1-1)$$

so that $\sum_i p_ix_i(\mathbf{p}, y) = y$ and budget balance is satisfied.

From the Slutsky equation, the elements S_{ij} of the substitution matrix are defined as

$$S_{ij} = \frac{\partial x_i}{\partial p_j} + x_j(\mathbf{p}, y)\frac{\partial x_i}{\partial y} \qquad (1-2)$$

So

$$S_{11} = -\frac{y}{(p_1)^2} + (0.5)(p_1)^{-1.5}(p_2)^{0.5} + (0.5)(p_1)^{-1.5}(p_3)^{0.5} + \frac{y}{(p_1)^2} - (p_1)^{-1.5}p_2^{0.5} - (p_1)^{-1.5}(p_3)^{0.5}$$

or

$$S_{11} = -(0.5)(p_1)^{-1.5}[(p_2)^{0.5} + (p_3)^{0.5}] < 0 \qquad (1-3)$$

$$S_{12} = -(0.5)(p_1p_2)^{-0.5} + (p_1)^{0.5}p_2^{-0.5}[\frac{1}{p_1}] = (0.5)(p_1p_2)^{-0.5} > 0 \qquad (1-4)$$

$$S_{13} = -(0.5)(p_1p_3)^{-0.5} + (p_1)^{0.5}p_3^{-0.5}[\frac{1}{p_1}] = (0.5)(p_1p_3)^{-0.5} > 0 \qquad (1-5)$$

$$S_{21} = (0.5)(p_1p_2)^{-0.5} > 0 \qquad (1-6)$$

$$S_{22} = -(0.5)(p_1)^{0.5}(p_2)^{-1.5} < 0 \qquad (1-7)$$

$$S_{23} = 0 \qquad (1-8)$$

$$S_{31} = (0.5)(p_1p_3)^{-0.5} > 0 \qquad (1-9)$$

$$S_{32} = 0 \quad (1-10)$$

$$S_{33} = -(0.5)(p_1)^{0.5}(p_3)^{-1.5} < 0 \quad (1-11)$$

Equations $(1-4)$ and $(1-6)$, $(1-5)$ and $(1-9)$, and $(1-8)$ and $(1-10)$ show that the substitution matrix is symmetric.

It remains to check whether that matrix is negative semi–definite. Equations $(1-3)$, $(1-7)$ and $(1-11)$ show that elements on the diagonal are negative. The determinant of the 2–by–2 matrix in the upper left–hand corner is

$$S_{11}S_{22} - S_{12}S_{21} = (0.25)[(p_1)^{-1}(p_2)^{-1.5}[(p_2)^{0.5} + (p_3)^{0.5}] - (p_1)^{-1}p_2^{-1}$$
$$=(0.25)[(p_1)^{-1}(p_2)^{-1.5}(p_3)^{0.5}] > 0$$

[So the second principal minor has a positive determinant.]

And the determinant of the whole matrix is

$$\det S = S_{11}S_{22}S_{33} - S_{33}[(S_{12})^2] - S_{22}[(S_{13})^2]$$

so that

$$8\det S = -(p_1)^{-0.5}(p_2)^{-1}(p_3)^{-1.5} - (p_1)^{-0.5}(p_2)^{-1.5}(p_3)^{-1}$$
$$+ (p_1)^{-0.5}(p_2)^{-1}(p_3)^{-1.5} + (p_1)^{-0.5}(p_2)^{-1.5}(p_3)^{-1}$$

which equals 0.

So the substitution matrix is negative semi–definite (since its three principal minors are negative, positive and zero in sign).

All 3 conditions in the Integrability Theorem are satisfied, so that the three functions could be Marshallian demand functions of a consumer with well–behaved preferences.

Q2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times. (For example, the third row indicates that the consumer chose the bundle $\mathbf{x} = (25, 15, 10)$ when the price vector was $\mathbf{p} = (3, 1, 1)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	3	20	20	10
2	2	1	2	30	10	20
3	3	1	1	25	15	10
4	1	3	1	20	15	20

A2. The following matrix displays the cost of each bundle at each year's prices.

year	$\mathbf{x}^1$	$\mathbf{x}^2$	$\mathbf{x}^3$	$\mathbf{x}^4$
year 1	70	100	70	95
year 2	80	110	85	95
year 3	90	120	100	95
year 4	90	80	80	85

(So, for example, the 85 in the 3rd column of the second row indicates that the bundle $\mathbf{x}^3$ chosen in year 3 would have cost \$85 in year 2, which is less than the cost of the bundle actually chosen in that year, which was \$110.)

Whenever a number on the diagonal of row i is greater than or equal to another element M_{ij} in that row, that means that $\mathbf{x}^i$ is revealed preferred (directly) to the bundle $\mathbf{x}^j$ chosen in year j.

The first row shows $\mathbf{x}^1$ is revealed preferred directly to $\mathbf{x}^3$.

The second row shows that $\mathbf{x}^2$ is revealed preferred directly to each of the other three bundles.

The third row shows that $\mathbf{x}^3$ is revealed preferred directly to $\mathbf{x}^1$ and $\mathbf{x}^4$.

The fourth row shows that $\mathbf{x}^4$ is revealed preferred directly to $\mathbf{x}^2$ and $\mathbf{x}^3$.

So there are 3 violations of $WARP$: year 1's bundle versus year 3's ; year 2's bundle versus year 4's ; year 3's bundle versus year 4's.

But, in a sense, every possible comparison violates $SARP$. For every one of the 6 pairs of years, there is a cycle.

For example, $\mathbf{x}^1$ is directly revealed preferred to $\mathbf{x}^3$ which is directly revealed preferred to $\mathbf{x}^4$ which is directly revealed preferred to $\mathbf{x}^2$ which is directly revealed preferred to $\mathbf{x}^1$. Or $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^1$ which is directly revealed preferred to $\mathbf{x}^4$ which is directly revealed preferred to $\mathbf{x}^2$. Or $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^3$ which is directly revealed preferred to $\mathbf{x}^4$ which is directly revealed preferred to $\mathbf{x}^2$.

Q3. If a risk–averse utility maximizer had a utility–of–wealth function

$$u(W) = \log W$$

what would be the risk premium to a gamble which would double her wealth with probability 0.5, and cut her wealth in half with probability 0.5?

A3. The certainty equivalent CE to the gamble is the solution to the equation

$$U(CE) = 0.5U(W/2) + 0.5U(2W) \qquad (3-1)$$

since she should be indifferent between getting CE for sure, and a gamble which gives her $W/2$ with probability 0.5 and $2W$ with probability 0.5. In this case, equation $(3-1)$ is

$$\log(CE) = 0.5\log(\frac{W}{2}) + 0.5\log(2W) \qquad (3-2)$$

Since $\log(ab) = \log a + \log b$, equation $(3-2)$ can be written

$$\log(CE) = 0.5[\log W + \log(\frac{1}{2}) + \log 2 + \log W] \qquad (3-3)$$

But $\log 2 + \log(\frac{1}{2}) = 0$, since $\log a + \log b = \log(ab)$, and since $\log 1 = 0$. So

$$\log CE = 0.5[2\log W] = \log W \qquad (3-4)$$

which means that $CE = W$.

The risk premium to the gamble is equal to the difference between the expected value Eg of her wealth (from the gamble), and the certainty equivalent.

Here

$$Eg = 0.5[\frac{W}{2} + 2W] = \frac{5W}{4} \qquad (3-5)$$

so that the risk premium equals

$$Eg - CE = \frac{W}{4}$$

Q4. Find a utility function $U(\cdot)$ such that the following statement would be true for an expected utility maximizer with this utility function : "If my initial wealth was 100, I would not make this investment, but if my initial wealth were 200 I would make the investment. The investment will decrease my wealth by 40 with probability 0.5, and increase my wealth by 50 with probability 0.5."

A4. Notice that this statement could be true for a person with a constant coefficient of relative risk aversion : any person with CRRA preferences would be more willing to undertake a risky investment, with given **absolute** size, if her initial wealth were higher.

So, if a person had a constant coefficient of relative risk aversion β, the statement would be true if

$$\frac{1}{1-\beta}(0.5)[(150)^{1-\beta} + (60)^{1-\beta}] < 2\frac{1}{1-\beta}[(100)^{1-\beta}] \qquad (4-1)$$

and if

$$\frac{1}{1-\beta}(0.5)[(250)^{1-\beta} + (160)^{1-\beta}] > 2\frac{1}{1-\beta}[(200)^{1-\beta}] \qquad (4-2)$$

Equation (1) holds whenever $\beta > 0.493558$, and equation (2) holds whenever $\beta < 1$.

So an example — but certainly not the only one — of a utility–of–wealth function which satisfies the 2 conditions is

$$U(W) = \frac{1}{1-\beta}W^{1-\beta}$$

for which $0.493558 < \beta < 1$.

Q5. How would the tax rate t on capital gains affect the following [risk–averse, von Neumann–Morgenstern expected utility maximizing] person's behaviour?

The person has an initial wealth of W. She chooses how much of that wealth to invest in a stock. The stock will either succeed or fail.

If the stock succeeds, she will make a gain of G dollars per dollar invested (so that if she had invested x dollars in the stock, the investment would now be worth $(G+1)x$ dollars.)

If the stock fails, it is worthless.

The probability of success for the stock is π.

Any money which she does **not** invest in the stock earns a net return of zero. (So that the $W - x$ dollars she does not invest in the stock will still be worth $W - x$, for certain.)

She must pay a tax of t dollars for every dollar of capital gains, but gets a tax credit of t dollars for every dollar she loses in the stock market.

A5. [This question is a version of a result obtained by Domar and Musgrave (QJE, 1944) on the effects of taxation on risk taking.]

If the tax rate is t, and if the person invests x dollars in the risky asset, then her expected utility will be

$$EU = \pi u[(G+1)x - tGx + W - x] + (1-\pi)U[W - x + tx] \qquad (5-1)$$

since she must pay taxes of tGx on her capital gains if the stock goes up, and she gets a tax credit of tx if her initial investment of x dollars becomes worthless.

She should choose the investment x in the risky asset to maximize expression $(5-1)$. So her first–order condition for this maximization (with respect to what she is choosing, the amount x to invest in the risky asset) is

$$\pi[G(1-t)]u'[Gx(1-t) + W] - (1-\pi)(1-t)u'[W - (1-t)x] = 0 \qquad (5-2)$$

Concavity of her utility–of–wealth function (she is risk–averse) ensures that the second–order conditions for utility maximization are satisfied : for a given tax rate t there is a unique x which solves equation $(5-2)$, and that x maximizes her expected utility.

Equation $(5-2)$ can be written

$$\pi G u'[Gx(1-t) + W] - (1-\pi)u'[W - (1-t)x] = 0 \qquad (5-3)$$

or

$$\pi G u'[Gz + W] - (1-\pi)u'(W - z) = 0 \qquad (5-4)$$

where

$$z \equiv (1-t)x$$

So let z^* be the (unique) solution to equation $(5-4)$. This z^* does **not** depend on the tax rate t : that tax rate does not appear in equation $(5-4)$. The person's actual optimal investment in the risky asset is

$$x^*(t) \equiv z^*/(1-t) \qquad (5-5)$$

What happens when the tax rate t increases? Equation $(5-5)$ says that increases in the tax rate lead to increases in x, enough so that the "net of tax" investment $(1-t)x$ stays constant (and equal to z^*).

So — when there is full "loss offset" — increases in the tax rate on the return to risky assets actually **increases** risk taking by investors.

By taxing the positive return, and offering a tax write–of for the negative return, the tax authority has cut itself in as a partner in the risky investment, and has made the (after–tax) return on the asset less risky. The investor responds to this “partnership” by increasing her own investment in the risky asset, so that her net–of–tax return (in either the good or the bad state) is unchanged.