*Q*1. For what input levels (x_1, x_2, x_3) does the following production function exhibit increasing returns to scale (using the "local" measure $\mu(\mathbf{x})$ of scale economies)?

$$f(\mathbf{x}) \equiv [x_1 x_2]^{\alpha} \frac{x_3}{1 + x_3}$$

where $\alpha > 0$.

*A*1. The definition of the local measure of scale economies $\mu(x)$ is that $\mu(\mathbf{x})$ is the sum of the elasticities

$$\mu_i(\mathbf{x}) \equiv \frac{\partial f}{\partial x_i} \frac{x_i}{f(\mathbf{x})}$$

of output with respect to input i.

Here

$$\mu_1(\mathbf{x}) = \alpha x_1^{\alpha-1} x_2^{\alpha} \frac{x_3}{1 + x_3} \frac{x_1}{f(\mathbf{x})} = \alpha \tag{1-1}$$

$$\mu_2(\mathbf{x}) = \alpha x_1^{\alpha} x_2^{\alpha-1} \frac{x_3}{1 + x_3} \frac{x_2}{f(\mathbf{x})} = \alpha \tag{1-2}$$

$$\mu_3(\mathbf{x}) = [x_1 x_2]^{\alpha} (1 + x_3)^{-2} \frac{x_3}{f(\mathbf{x})} = \frac{1}{1 + x_3} \tag{1-3}$$

So that

$$\mu(\mathbf{x}) = 2\alpha + \frac{1}{1 + x_3} \tag{1-4}$$

If $\alpha \geq 0.5$, then the function exhibits increasing returns to scale everywhere.

If $\alpha < 0.5$, then the function exhibits increasing returns whenever

$$2\alpha + \frac{1}{1 + x_3} > 1 \tag{1-5}$$

which is equivalent to

$$x_3 < \frac{2\alpha}{1 - 2\alpha} \tag{1-6}$$

If $\alpha < 0.5$, and if $x_3 > \frac{2\alpha}{1-2\alpha}$, then the function exhibits decreasing returns to scale.

*Q*2. Derive the cost function for the production function

$$f(x_1, x_2) = \log(x_1 + 1) + \log(x_2 + 1) - \log(x_1 + x_2 + 2) + \log 2$$

*A*2. This cost function is pretty similar to the utility function from question #5 in assignment 1 from F2011. The one difference is that x_1 and x_2 have been replaced by $x_1 + 1$ and $x_2 + 1$ (and a constant has been added) to ensure that $f(0, 0) = 0$.

So the conditional input demand functions in this question are going to be the Hicksian demand functions from that other question, minus 1.

Formally, minimization of $w_1 x_1 + w_2 x_2$ subject to the constraint that $f(x_1, x_2) \geq y$ yields first–order conditions

$$\frac{1}{x_1 + 1} - \frac{1}{x_1 + x_2 + 2} = \mu w_1 \tag{2-1}$$

$$\frac{1}{x_2 + 1} - \frac{1}{x_1 + x_2 + 2} = \mu w_2 \tag{2-2}$$

which can be written

$$\frac{1}{X_1} - \frac{1}{X_1 + X_2} = \mu w_1 \tag{2-3}$$

$$\frac{1}{X_2} - \frac{1}{X_1 + X_2} = \mu w_2 \tag{2-4}$$

if we let $X_i \equiv x_i + 1$. These two equations can be written

$$\frac{X_2}{X_1(X_1 + X_2)} = \mu w_1 \tag{2-5}$$

$$\frac{X_1}{X_2(X_1 + X_2)} = \mu w_2 \tag{2-6}$$

so that

$$[\frac{X_2}{X_1}]^2 = \frac{w_1}{w_2} \tag{2-7}$$

or

$$X_2 = \sqrt{\frac{w_1}{w_2}} X_1 \tag{2-8}$$

The constraint that $f(x_1, x_2) = y$ is

$$\log X_1 + \log X_2 - \log (X_1 + X_2) = y - \log 2 \tag{2-9}$$

so that substitution of $(2-8)$ into $(2-9)$ yields

$$\log X_1 + \log \sqrt{w_1} - \log \sqrt{w_2} + \log X_1 - \log (\sqrt{w_1} + \sqrt{w_2}) - \log X_1 + \log \sqrt{w_2} = y - \log 2 \tag{2-10}$$

or

$$X_1 = e^{(y - \log 2)} \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_1}} \tag{2-11}$$

or

$$x_1(w_1, w_2, y) = \frac{1}{2} e^y \frac{\sqrt{w_1} + \sqrt{w_1}}{\sqrt{w_1}} - 1 \tag{2-12}$$

which is the conditional input demand function for input 1. From $(2-8)$, then

$$x_2(w_1, w_2, y) = \frac{1}{2} e^y \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_2}} - 1 \tag{2-13}$$

The cost function is $w_1x_1(w_1,w_2,y)+w_2x_2(w_1,w_2,y)$, or (from equations $(2-12)$ and $(2-13)$)

$$C(w_1,w_2,y)=\frac{1}{2}e^y[\sqrt{w_1}+\sqrt{w_2}]^2-w_1-w_2 \tag{2-14}$$

[If one input is much cheaper than the other, then there may be a corner solution, if the output level y is small.

Expressions $(2-12)$ and $(2-13)$ imply that both x_1 and x_2 must be non–negative, no matter what the input prices w_1 and w_2, if $e^y \geq 2$.

But if $y < \log 2$, then expression $(2-12)$ will be negative if (and only if)

$$\sqrt{\frac{w_2}{w_1}} < 2e^{-y}-1 \tag{2-15}$$

If inequality $(2-15)$ holds, then we have a corner solution, in which $x_1=0$. The value of x_2 then, is the solution to $f(0,x_2)=y$, or

$$x_2(w_1,w_2,y)=2\frac{e^y-1}{2-e^y} \tag{2-16}$$

so that

$$C(w_1,w_2,y)=2w_2\frac{e^y-1}{2-e^y} \tag{2-17}$$

Conversely, if $\sqrt{\frac{w_1}{w_2}} < 2e^{-y}-1$, then $x_2=0$, and

$$C(w_1,w_2,y)=2w_1\frac{e^y-1}{2-e^y} \qquad (2-18) \qquad]$$

*Q*3. What are total industry profits (as a function of the price p in long–run equilibrium in the following perfectly competitive industry?

A type–t firm has a total cost function, as a function of the firm's output q of

$$C_i(q)=\frac{1}{2}q^2+At$$

where A is a positive constant.

There is a continuum of firms. Firms differ in their type $t:$ this type is distributed uniformly over $[0,1]$. The total "number" of firms (that is the measure of all firms of all types) is some finite M.

[So half the firms have a value of t between 0 and 0.5, a quarter of the firms have a value of t between 0 and 0.25, etc.]

*A*3. For each firm, whatever its type,

$$MC(q)=q \tag{3-1}$$

so that each firm's marginal cost curve slopes up. For any firm of type t, then, if it chooses to enter the industry, it will choose a quantity of output $q(t)$ such that $MC(q(t)) = p$, or (from equation $(3-1)$)

$$q(t) = p \qquad (3-2)$$

The firm's total profit, if it chooses to enter, is

$$\pi(p,t) = pq(t) - \frac{1}{2}(q(t))^2 - At \qquad (3-3)$$

or (from equation $(3-2)$)

$$\pi(p,t) = \frac{p^2}{2} - At \qquad (3-4)$$

Equation $(3-4)$ confirms that a firm's profit decreases with its type t, since high–t firms have higher fixed costs.

So the highest type T of firm is the type $T(p)$ for which $\pi(p,t) = 0$, or

$$T(p) = \frac{p^2}{2A} \qquad (3-5)$$

which is a non–decreasing function of the price p.

The number of firms of type between 0 and t is Mt, because of the assumption that firms' types are distributed uniformly over $[0,1]$.

If $T(p) < 1$, then total industry profit is

$$\Pi(p) = M[\int_0^{T(p)} \pi(p,t)dt] = M[\int_0^{T(p)} [\frac{p^2}{2} - At]dt] \qquad (3-6)$$

or (from the definition $(3-5)$ of $T(p)$)

$$\Pi(p) = \frac{Mp^4}{8A} \qquad (3-7)$$

Equation $(3-7)$ holds only if $T(p) \leq 1$. If $T(p) > 1$, then there is no entry or exit at the margin. From equation $(3-5)$, $T(p) = 1$ when

$$p = \sqrt{2A} \qquad (3-8)$$

When $p > \sqrt{A}$ there is no further entry of new firms, so that

$$\Pi(p) = M[\int_0^1 [\frac{p^2}{2} - At]dt] \qquad (3-9)$$

or

$$\Pi(p) = \frac{M(p^2 - A)}{2} \qquad (3-10)$$

$Q4$. What is the equilibrium when two firms choose quantities simultaneously (that is, in a Cournot duopoly) if the market demand function for the firms' identical products was

$$p = A - Q$$

where $Q \equiv q_1 + q_2$ is the combined output of the two firms, when firm 1's total cost function is

$$TC_1(q_1) = q_1$$

and firm 2's total cost function is

$$TC_2(q_2) = q_2 + 4 \quad \text{if} \quad q_2 > 0 \qquad ; \qquad TC_2(0) = 0$$

where A is some positive constant? [So firm 2 has a fixed cost of 4, which it can avoid only by producing nothing.]

*A*4. Each firm has a constant marginal cost of 1. So, if each firm chooses to produce a positive quantity of output, it has a reaction function

$$q_i = \frac{A-1}{2} - \frac{q_j}{2} \tag{4-1}$$

where j is the other firm's output.

So the only possible equilibrium output levels (q_1, q_2) for which both q_1 and q_2 are positive is

$$q_1^{eq} = q_2^{eq} = \frac{A-1}{3} \tag{4-2}$$

If $(4-2)$ holds for both firms, then total industry output is

$$Q \equiv q_1 + q_2 = \frac{2(A-1)}{3} \tag{4-3}$$

so that

$$p = \frac{A+2}{3} \tag{4-4}$$

meaning that firm 1 earns profits of

$$\pi_1 = (p-1)q_1 = \frac{(A-1)^2}{9} \tag{4-5}$$

and firm 2 earns profits of

$$\pi_2 = (p-1)q_2 - 4 = \frac{(A-1)^2}{9} - 4 \tag{4-6}$$

Note that expression $(4-5)$ is always positive, no matter what the value of A. But expression $4-6)$ is only positive if $A > 7$.

So if $A > 7$, there is a "standard" Cournot equilibrium, in which both firms produce the same positive level of output $q_1 = q_2 = \frac{(A-1)}{3}$. But if $A < 7$, then firm 2 cannot make a positive profit. Hence, if $A < 7$, firm 2 chooses to produce no output at all, and firm 1 reacts to $q_2 = 0$ by producing its best reaction (from equation $(4-1)$),

$$q_1^{mono} = \frac{A-1}{2} \tag{4-7}$$

But even if A is (a little) above 7, there may be an equilibrium in which firm 2 produces nothing.

Suppose that firm 1 produces the monopoly output. From equation $(4-1)$, firm 2's best reaction to $q_1 = \frac{A-1}{2}$ — if it were to produce a positive level of output — is

$$q_2^r = \frac{A-1}{4} \tag{4-8}$$

If firm 1 produces q_1^{mono} and firm 2 produces q_2^r, then

$$p = \frac{A+3}{4} \tag{4-9}$$

so that firm 2's profit would be

$$\pi_2^r = \frac{(A-1)^2}{16} - 4 \tag{4-10}$$

which is positive only if $A > 9$.

So the best reaction by firm 2 to $q_1 = q_1^{mono} = \frac{A-1}{2}$ is to produce nothing at all, if $A < 9$.

That means that $q_1 = \frac{A-1}{2}, q_2 = 0$ is an equilibrium in the Cournot duopoly, if $A \leq 9$. Firm 2 responds to the monopoly output (of firm 1) by shutting down, and firm 1 responds to firm 2 shutting down by producing the monopoly output.

So when $7 \leq A \leq 9$ there are actually 2 different equilibria in this Cournot duopoly. One has both firm's producing the "standard" Cournot outputs $q_1 = q_2 = \frac{A-1}{3}$, and the other has $q_1 = \frac{A-1}{2}$ and $q_2 = 0$. When $A > 9$, only the first pair of outputs is an equilibrium, and when $A < 7$ only the second pair is an equilibrium.

*Q*5. Find a symmetric Bertrand equilibrium, when two firms produce goods which are close (but imperfect) substitutes, with each firm i facing a demand curve

$$q_i = \frac{p_i^{-\alpha-1}}{p_i^{-\alpha} + p_j^{-\alpha}}$$

where p_j is the other firm's price, when each firm has a constant marginal cost c.

The parameter α is positive (and equals $\sigma - 1$, where σ is buyers' elasticity of substitution between the 2 goods).

*A*5. Given the demand function, firm #1's profit, $p_1 q_1 - c q_1$, equals

$$\pi_1 = \frac{p_1^{-\alpha}}{p_1^{-\alpha} + p_2^{-\alpha}} - \frac{c p_1^{-\alpha-1}}{p_1^{-\alpha} + p_2^{-\alpha}} \tag{5-1}$$

As a Bertrand competitor, firm #1 chooses p_1 so as to maximize expression $(5-1)$, taking as given the other firm's price p_2. The first–order condition for this maximization is

$$\frac{\partial \pi_1}{\partial p_1} = \frac{p_1^{-\alpha-2}}{[p_1^{-\alpha} + p_2^{-\alpha}]^2}[-\alpha p_1 p_2^{-\alpha} + c p_1^{-\alpha} + (\alpha+1) c p_2^{-\alpha}] = 0 \tag{5-2}$$

So firm #1's reaction to firm #2's price p_2 can be written as

$$-\alpha p_1 p_2^{-\alpha} + c p_1^{-\alpha} + (\alpha+1) c p_2^{-\alpha} = 0 \tag{5-3}$$

(Differentiating equation $(5-3)$ with respect to p_1 and p_2 would give the slope of firm #1's reaction function in p_2–p_1 space.)

Firm #2 has an analogous problem, and a reaction function

$$-\alpha p_2 p_1^{-\alpha} + c p_2^{-\alpha} + (\alpha+1) c p_1^{-\alpha} = 0 \tag{5-4}$$

If there is a symmetric equilibrium, the common price p chosen by the two firms will be the solution to $(5-3)$ (or $(5-4)$) with $p_1 = p_2 = p$, or

$$-\alpha p^{1-\alpha} + c p_1^{-\alpha} + (\alpha+1) c p^{-\alpha} = 0 \tag{5-5}$$

meaning that the common price p is

$$p = \frac{\alpha+2}{\alpha} c \tag{5-6}$$

So the equilibrium price p exceeds the firms' common marginal cost. As long as the two firms' products are less–than–perfect substitutes, Bertrand competition does not drive their profits down to zero. Notice that the higher the elasticity of substitution $\alpha + 1$, the lower is the price. As $\alpha \to \infty$, then $p \to c$, so that the "standard" Bertrand result, of undercutting driving the price down to marginal cost, is a special case here, as the two firms' products approach being perfect substitutes for each other.