GS/ECON 5010   Answers to Assignment #4   November 2017

*Q*1. What does the contract curve look like for a 2–person exchange economy, in which the preferences of the two people can be represented by the utility functions

$$U^1(x_1^1, x_2^1) = \log(x_1^1) + x_2^1$$

$$U^2(x_1^2, x_2^2) = \log(x_1^2) + \log(x_2^2) \qquad ?$$

*A*1. If an allocation $\mathbf{x}^1, \mathbf{x}^2$, with $\mathbf{x}^1 >> 0$ and $\mathbf{x}^2 >> 0$, is Pareto efficient, then it must be true that the two people's MRS's are the same.

With the preferences in the question, $MRS^1 = MRS^2$ if and only if

$$\frac{1}{x_1^1} = \frac{x_2^2}{x_1^2} \tag{1-1}$$

Since $x_i^1 + x_i^2 = E_i$ for good i, where E_i is the total endowment of good i, then equation $(1-1)$ can be written

$$\frac{1}{x_1^1} = \frac{E_2 - x_2^1}{E_1 - x_1^1} \tag{1-2}$$

or

$$x_2^1 = E_2 + 1 - \frac{E_1}{x_1^1} \tag{1-3}$$

Equation $(1-3)$ is the equation of the contract curve, since it expresses person 1's consumption of good 2 as a function of her consumption of good 1. Since $(1-3)$ implies that

$$\frac{dx_1^2}{dx_1^1} = \frac{E_1}{(X_1^1)^2} \tag{1-4}$$

the curve is upward–sloping, and concave.

When person #1 has all of good 1 — $x_1^1 = E_1$ — then equation $(1-3)$ implies that $x_2^1 = E_2$. So the contract curve goes through the top right corner of the Edgeworth box.

But the curve does not go through the bottom–left corner of the Edgeworth box. If person #2 were to get all of the total endowment of good #1, so that $x_1^1 = 0$, then equation $(1-3)$ implies $x_2^1 \to -\infty$, an impossibility.

Due to person 1's quasi–linear preferences, the contract curve hits the bottom of the Edgeworth box to the right of the origin. Equation $(1-3)$ says that $x_2^1 = 0$ when

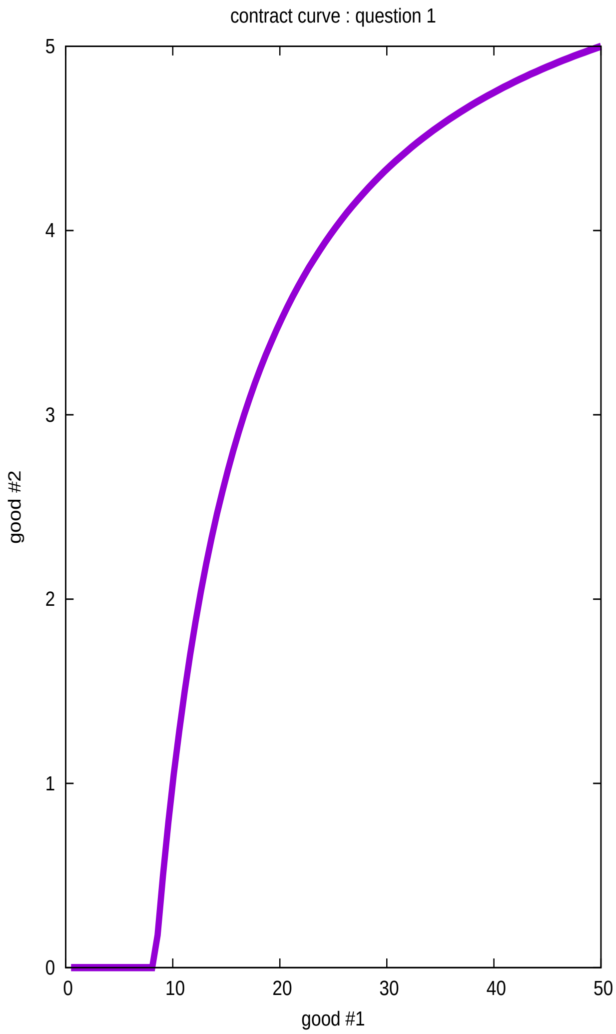
$$x_1^1 = \frac{E_1}{E_2 + 1} \tag{1-5}$$

So the point

$$(x_1^1, x_2^1) = (\frac{E_1}{E_2 + 1}, 0)$$

is on the contract curve.

That means that, for these preferences, the contract curve has two pieces : it is the bottom of the Edgeworth box, from the point $(0, 0)$ to the point $(\frac{E_1}{E_2+1}, 0)$, and then it is the upward–sloping curve defined by equation $(1-3)$.



contract curve : question 1
good #2
good #1
0
1
2
3
4
5
0
10
20
30
40
50

$Q2$. Show that the following allocation is not in the core of the 20–person exchange economy described below. (That is, find a coalition which **blocks** the allocation.)

The allocation is

$$\mathbf{x}^i = (76, 76) \quad \text{for} \quad i = 1, 2, 3, \cdots, 9$$

$$\mathbf{x}^{10} = (66, 66)$$

$$\mathbf{x}^i = (125, 125) \quad \text{for} \quad i = 11, 12, \cdots, 20$$

In this economy, the preferences of each of the 20 people can be represented by the utility function

$$u^i(x_1^i, x_2^i) = \log(x_1^i) + \log(x_2^i)$$

and the endowments are

$$\mathbf{e}^i = (150, 0) \quad \text{for} \quad i = 1, 2, \cdots, 10$$

$$\mathbf{e}^i = (50, 200) \quad \text{for} \quad i = 11, 12, \cdots, 20$$

$A2$. Since all people have identical, homothetic preferences, an allocation is Pareto efficient only if x_1^i/x_2^i is equal across all people i. In the allocation $\mathbf{x}$, $x_1^i = x_2^i$ for each person, so that the allocation is Pareto optimal, and cannot be blocked by a "coalition of everybody".

Each allocation in the core must give each person higher utility than her endowment : otherwise it could be blocked by a coalition consisting of one person. Here the first 10 people get utility of 0 from consuming their own endowment, and positive utility from the allocation $\mathbf{x}$. The last 10 people get utility of $(125)(125) = 15625 > (50)(200) = 10000$, so that everyone prefers $\mathbf{x}$ to her or his own endowment : the allocation cannot be blocked by a coalition of 1 person.

But it can be blocked by a coalition of 2 people. Person 10 is an obvious candidate for someone to join a blocking coalition. If she forms a coalition with any one person j, with $j > 10$, this 2–person coalition has a total endowment of $(200, 200)$. So, for example, person 10 and person 11 could form a 2–person coalition and provide the consumption bundles $\mathbf{y}^{11} = (70, 70), \mathbf{y}^{11} = (130, 130)$ to its 2 members. Since $\mathbf{y}^{10} >> \mathbf{x}^{10}$ and $\mathbf{y}^{11} >> \mathbf{x}^{11}$, both person 10 and person 11 would prefer to form this coalition, over accepting the allocation $\mathbf{x}$. Therefore, the coalition $S = \{10, 11\}$ blocks the allocation $\mathbf{x}$ with the allocation $\mathbf{y}^{10} = (70, 70), \mathbf{y}^{11} = (130, 130)$.

(This is certainly not the only coalition which can block $\mathbf{x}$.)

$Q3$. Find all the allocations in the **core** of the following 3–person economy.

Each person has the same preferences : person i's preferences can be represented by the utility function

$$u^i(x_1^i, x_2^i) = x_1^i x_2^i \qquad i = 1, 2, 3$$

The endowment vectors $\mathbf{e}^i$ of the three people are

$$\mathbf{e}^1 = (3, 0)$$

$$\mathbf{e}^2 = (0, 3)$$

$$\mathbf{e}^3 = (1, 1)$$

$A3$. Every allocation in the core must be Pareto optimal (but not vice versa) : if an allocation were not Pareto optimal, then it could be blocked by the coalition of everybody, by finding an allocation which is Pareto–preferred too it.

Here each person's marginal rate of substitution is

$$MRS^i \equiv \frac{U_1^i}{U_2^i} = \frac{x_2^i}{x_1^i} \qquad (3-1)$$

so that the Pareto optimality condiiton $MRS^1 = MRS^2 = MRS^3$ requires

$$x_2^1/x_1^1 = x_2^2/x_1^2 = x_2^3/x_1^3 \qquad (3-2)$$

Since the total endowments of each good are equal, condition $(3-2)$ requires that $x_1^i = x_2^i$ for each person i.

That means that an allocation in the core must be of the form

$$\mathbf{x}^1 = (a, a)$$

$$\mathbf{x}^2 = (b, b)$$

$$\mathbf{x}^3 = (c, c)$$

for some positive a b and c, with $a + b + c = 4$.

Person #3 gets utility 1 from consuming her own endowment. Therefore it must be true that $c \geq 1$ for any allocation in the core : if $c < 1$ a coalition–of–one, of person #3 would block the allocation by consuming her own endowment $(1, 1)$.

A coalition of person 1 and 2 has a total endowment of $(3, 3)$. So that coalition can guarantee its two members consumption bundles $\mathbf{x}^1 = (a, a)$ and $\mathbf{x}^2 = (3 - a, 3 - a)$, for any $0 \leq a \leq 3$. That means that any allocation in which $c > 1$ can be blocked. If $c > 1$, then $b = 4 - a - c < 3 - a$, so that the coalition of $\{1, 2\}$ can give person #1 (a, a) and person #2 $(3 - a, 3 - a) >> (b, b)$, and block the allocation.

So an allocation in the core must therefore be of the form

$$\mathbf{x}^1 = (a, a)$$

$$\mathbf{x}^2 = (3 - a, 3 - a)$$

$$\mathbf{x}^3 = (1, 1)$$

if it is in the core.

The remaining chore is to check what values of a cannot be blocked.

So suppose that a is small. Person #1 would like to block this allocation by forming a coalition with person #3 : note than she cannot block the allocation by forming a coalition with person #2 (since that coalition cannot do better than $\mathbf{x}^1 = (a, a)$ and $\mathbf{x}^2 = (3-a, 3-a)$). To get person #3 to join the coalition, person #1 must offer her a utility at least as high as she gets under the proposed allocation : 1.

The coalition of #1 and #3 has a total endowment of $(4, 1)$. So to get person #3 to join the coalition, and to block the original allocation, person #1 must solved the problem of finding $\mathbf{y}^1$ and $\mathbf{y}^3$ for people #1 and #3 so as to maximize her own ultility $(y_1^1)(y_2^1)$ subject to the constraints that $y_1^1 + y_1^3 = 4$, $y_2^1 + y_2^3 = 1$ and the constraint $(y_1^3)(y_2^3) \geq 1$ that is needed to get person #3 to join.

Substituting from the resource constraints $y_1^1 = 4 - y_1^3$ and $y_2^1 = 1 - y_2^3$, person #1 therefore picks y_1^3 and y_2^3 to maximize

$$(4 - y_1^3)(1 - y_2^3)$$

subject to the constraint that

$$(y_1^3)(y_2^3) = 1$$

Solving this maximization (either setting up a Lagrangean, or substitution from the constraint into the maximand) implies that

$$y_1^3 = 2 \tag{3-3}$$

$$y_2^3 = 0.5 \tag{3-4}$$

[Why does this make sense? if person #1 and person #3 form a blocking coalition, they should ensure that their coalition allocates its available resources $(4, 1)$ efficiently. Efficiency here means ensuring that the available quantities of the two goods be divided so that $MRS_1 = MRS_3$. With identical homothetic preferences, that means that we should have $y_1^3 = 4y_2^3$ when the aggregate endowment of the coalition is $(4, 1)$.]

So if person #1 does not like her allocation (a, a), the only thing she can do to block it is to form a coalition with person #3, and give person #3 the allocation $(2, 0.5)$, which leaves person #1 with the consumption bundle

$$\mathbf{y}^1 = (2, 0.5)$$

and a utility level of 1.

That means an allocation in which $a < 1$ can be blocked by the coalition of #1 and #3, with the allocation $\mathbf{y}^1 = \mathbf{y}^2 = (2, 0.5)$. If $a > 1$, then the allocation cannot be blocked.

Analogously, if $b < 1$, the allocation can be blocked by a coalition of person #2 and person #3, but if $b > 1$ it cannot be blocked by this coalition.

So the core of the economy is all allocations of the form

$$\mathbf{x}^1 = (a, a)$$

$$\mathbf{x}^2 = (3-a, 3-a)$$

$$\mathbf{x}^3 = (1, 1)$$

provided that $1 \leq a \leq 2$.

[So the allocation $(1.2, 1.2), (1.8, 1.8), (1, 1)$ is in the core, but the allocation $(0.8, 0.8), (2.2, 2.2), (1, 1)$ is not.]

$Q4$. Find a competitive equilibrium price vector for the following exchange economy.
There are 3 million people in the economy.
Each of the three million people has the same endowment vector,

$$\mathbf{e}^i = (e_1, e_2, e_3)$$

One million people are "type 1" people, and have preferences represented by the utility function

$$u^i(\mathbf{x}^i) = x_1^i x_2^i x_3^i$$

One million people are "type 2" people, and have preferences represented by the utility function

$$u^i(\mathbf{x}^i) = x_2^i$$

One million people are "type 3" people, and have preferences represented by the utility function

$$u^i(\mathbf{x})^i = (x_1^i)(x_3^i)^2$$

$A4$. Each person's endowment is worth

$$y = p_1 e_1 + p_2 e_2 + p_3 e_3 \tag{4-1}$$

if the market price vector is $\mathbf{p} \equiv (p_1, p_2, p_3)$.

Given the Cobb–Douglas preferences for person #1 and person #3, and person #2's preference for consuming only good #2, the Marshallian demand functions of the three people are

$$x_1^1(\mathbf{p}, y) = \frac{y}{3p_1} \tag{4-2}$$

$$x_2^1(\mathbf{p}, y) = \frac{y}{3p_2} \tag{4-3}$$

$$x_3^1(\mathbf{p}, y) = \frac{y}{3p_3} \tag{4-4}$$

$$x_2^2(\mathbf{p}, y) = \frac{y}{p_2} \tag{4-5}$$

$$x_3^1(\mathbf{p}, y) = \frac{y}{3p_1} \tag{4-6}$$

$$x_3^3(\mathbf{p}, y) = \frac{2y}{3p_3} \tag{4-7}$$

(with the left–out demands all being 0). In equilibrium, total quantity demanded of each good must equal the total endowment, so that the market–clearing conditions for the three goods are

$$\frac{y}{3p_1} + \frac{y}{3p_1} = 3e_1 \tag{4-8}$$

$$\frac{y}{3p_2} + \frac{y}{p_2} = 3e_2 \tag{4-9}$$

$$\frac{y}{3p_3} + \frac{2y}{3p_3} = 3e_3 \tag{4-10}$$

These equations can be written

$$2y = 9e_1 p_1 \tag{4-11}$$

$$4y = 9e_2 p_2 \tag{4-12}$$

$$5y = 9e_3p_3 \tag{4-13}$$

so that

$$\frac{p_2}{p_1} = 2\frac{e_1}{e_2} \tag{4-14}$$

$$\frac{p_3}{p_1} = \frac{3}{2}\frac{e_1}{e_3} \tag{4-15}$$

Equations $(4-14)$ and $(4-15)$ actually define the equilibrium prices, as functions of the relative endowments of the 3 goods. That is, an equilibrium price vector is any vector (p_1, p_2, p_3), in which p_1 can be any positive number, and the other two prices p_2 and p_3 are defined by equations $(4-14)$ and $(4-15)$.

For example

$$\mathbf{p} = (\frac{1}{e_1}, \frac{2}{e_2}, \frac{1.5}{e_3}) \tag{4-16}$$

is an equilibrium price vector. Here the equilibrium price of a good is a decreasing function of the endowment of the good.

The question did not ask for the equilibrium consumption bundles for the consumers, but substitution from $(4-16)$ into $(4-2)$–$(4-7)$ yields equilibrium consumption levels of

$$\mathbf{x}^1 = (\frac{3e_1}{2}, \frac{3e_2}{4}, e_3) \tag{4-17}$$

$$\mathbf{x}^2 = (0, \frac{9e_2}{4}, 0) \tag{4-18}$$

$$\mathbf{x}^3 = (\frac{3e_1}{2}, 0, 2e_3) \tag{4-19}$$

Q5. Find all the Nash equilibria (in pure and mixed strategies) to the following two–person game in strategic form.

	L	M	R
a	$(2,2)$	$(10,1)$	$(2,6)$
b	$(6,4)$	$(12,3)$	$(2,12)$
c	$(0,12)$	$(10,10)$	$(1,10)$
d	$(12,2)$	$(6,0)$	$(0,0)$

A5. Notice first that strategy c for player #1 is strictly dominated by strategy b : that means that player #1 will never play strategy c, and will never put any weight on strategy c when she chooses a mixed strategy.

Also, strategy M is strictly dominated for player 2 by strategy L. So, again player #2 will never play strategy M, and will never put any weight on strategy M when he chooses a mixed strategy.

Strategy a is weakly dominated by strategy b for player #1. She might still play strategy a in equilibrium — but she will never put any probability weight on strategy a in a mixed–strategy equilibrium.

There are three pure strategy Nash equilibrium to this game : (a,R), (b,R), and (d,L). Even though a is a weakly dominated strategy, neither player has any incentive to change her or his strategy when player #1 plays a and player #2 plays R.

Since a and b are both best responses (for player #1) to R, and since R is a best response (for player #2) to a and to b, there are "partially mixed" strategy equilibria in which player #1 randomizes between a and b, and player #2 plays R (for sure) : if player #1 plays a with probability γ, and b with probability $1-\gamma$ — for any γ in $[0,1]$ — then there is a Nash equilibrium, in which player #2 plays R for certain, and player #1 randomizes between a and b.

Finally, if player #2 were to randomize between L and R, player #1 could not choose a with positive probability (since it is weakly dominated). Player 1 would get an expected payoff of $6\beta+2(1-\beta)$ from playing b, and an expected payoff of 12β from playing d, if player #2 were to play L with probability β and R with probability $1-\beta$. So player #1 would be willing to randomize between b and d only if

$$6\beta+2(1-\beta)=12\beta \qquad (5-1)$$

or

$$\beta=\frac{1}{4}$$

On the other hand, if player #1 played b with probability α and d with probability $1-\alpha$, then player #2 would be willing to randomize between L and R, if they both offered him the same expected payoff. Since he would get an expected payoff of $4\alpha+2(1-\alpha)$ from L, and 12α from R, he will be willing to randomize only if

$$4\alpha+2(1-\alpha)=12\alpha \qquad (5-2)$$

or

$$\alpha=\frac{1}{5}$$

So there is a mixed–strategy equilibrium in which player #1 plays b with probability 0.2, d with probability 0.8, and the other two strategies with probability 0, and in which player #2 plays L with probability 0.25, M with probability 0, and R with probability 0.75.

Summarizing, in this game there are

(a) 3 pure–strategy Nash equilibria : (a,R), (b,R), and (d,L)

(b) a mixed–strategy Nash equilibrium in which 1 plays a with probability γ, b with probability $1-\gamma$, and 2 plays R for sure [where $0\leq\gamma\leq1$]

(c) a mixed–strategy Nash equilibrium in which 1 plays b with probability 0.2 and d with probability 0.8, and 2 plays L with probability 0.25 and R with probability 0.75