GS/ECON 5010 Answers to Assignment 1 February 2005

1. Q Are the preferences represented by the following utility function strictly monotonic? Convex?

$$u(x_1, x_2) = (x_1)^2 + (x_2)^2$$

In each case, explain briefly.

A : Since this utility function is strictly increasing in both arguments, the preferences it represents are strictly monotonic.

But the preferences are not convex. Consider the following two consumption bundles : $\mathbf{x}^1 = (1, 0)$ and $\mathbf{x}^2 = (0, 1)$. Both bundles are on the same indifference curve, since they each yield a utility level of 1. But the bundle $\mathbf{x}^3$ which is halfway along the line connecting the two bundles,

$$\mathbf{x}^3 = (0.5)\mathbf{x}^1 + (0.5)\mathbf{x}^2 = (\frac{1}{2}, \frac{1}{2})$$

yields a utility level of

$$u(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1$$

So the "at least as good as" sets, $\succeq (\mathbf{x})$, are not convex : $\mathbf{x}^1 \in \succeq (\mathbf{x})$, $\mathbf{x}^2 \in \succeq (\mathbf{x})$, but $\mathbf{x}^3 = (0.5)\mathbf{x}^1 + (0.5)\mathbf{x}^2 \notin \succeq (\mathbf{x})$, if, for example $\mathbf{x} = (0.7, 0.7)$.

Alternatively, one could look at the shape of the indifference curves. The equation of an indifference curve for these preferences is

$$x_2 = \sqrt{A - (x_1)^2}$$

for some level of utility A, and differentiation of this equation shows that the slope of the indifference curve gets more steep as x_1 increases and x_2 decreases.

Thirdly, the matrix of second derivatives of the utility function is

$$H(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

which is a positive definite matrix, meaning that the function $U(\mathbf{x})$ is strictly convex : a strictly convex function cannot be quasi–concave. [This matrix can also be used to show that $\mathbf{v}'H(\mathbf{x})\mathbf{v} > 0$ when $\nabla U(\mathbf{x}) \cdot \mathbf{v} = \mathbf{0}$, and it also can be showed that the bordered Hessian matrix does not have the alternating sign pattern required for quasi–concavity..]

2. Q Are the preferences represented by the following utility function strictly monotonic? Convex?

$$u(x_1, x_2, x_3) = x_1 x_2 + x_3$$

In each case, explain briefly.

A Since all the partial derivatives of this function are non–negative, and since the partial derivative with respect to commodity #3 is strictly positive, therefore preferences are strictly monotonic.

One way of checking for convexity of the preferences is to calculate the matrix $H(\mathbf{x})$ of second derivatives of the utility function. That matrix is

$$H(\mathbf{x}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So that

$$\mathbf{v}' H(\mathbf{x}) \mathbf{v} = 2 v_1 v_2$$

Now if

$$\nabla u(\mathbf{x}) \cdot \mathbf{v} = 0$$

then

$$v_1 x_2 + v_2 x_1 + v_3 = 0$$

If $\mathbf{x} = (1, 1, 1)$, and if $\mathbf{v} = (1, 1, -2)$, then $\mathbf{v}' H(\mathbf{x}) \mathbf{v} = 2 > 0$, even though $\nabla U(\mathbf{x}) \cdot \mathbf{v} = 0$. So preferences here are **not** convex.

Another way to demonstrate that these preferences are not convex, is to come up with a counter–example. That is, if there is some pair of consumption vectors $\mathbf{x}^1$ and $\mathbf{x}^2$, with $u(\mathbf{x}^1) = u(\mathbf{x}^2)$, and some fraction t between 0 and 1, such that

$$u(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) < u(\mathbf{x}^1)$$

then the preferences have been demonstrated not to be convex.

Here is one such example.

$u(1, 1, 10) = 11$, and $u(3, 3, 2) = 11$. But if we take the consumption bundle which is halfway between these bundles, then $u(2, 2, 6) = 10 < 11$. So this is an example of two bundles which are on the same indifference surface, and of another bundle which is on the line connecting them, which is on a lower indifference surface.

The bordered Hessian test will also work here. The bordered Hessian matrix is

$$\begin{pmatrix} 0 & x_2 & x_1 & 1 \\ x_2 & 0 & 1 & 0 \\ x_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The determinants of the principal minors of order 2 or more [cf. *Jehle and Reny*, page 511] are $M_2 = -x_2^2 < 0$, $M_3 = 2x_1 x_2 > 0$, and $M_4 = 1 > 0$, so that they do not have the alternating sign pattern required for quasi–concavity.

3. Q Solve for a person's Marshallian demand functions, if her preferences can be represented by the utility function

$$u(x_1, x_2) = -\exp(-x_1) - \exp(-x_2)$$

(where "exp" is the exponential function, $\exp a \equiv e^a$).

A Since the derivative of e^x with respect to x is e^x, it follows that

$$u_1 = \exp(-x_1)$$

$$u_2 = \exp(-x_2)$$

so that the marginal rate of substitution is

$$MRS \equiv u_2/u_1 = \exp(x_1 - x_2)$$

where I have used the fact that

$$\frac{e^a}{e^b} = e^{a-b}$$

Since the consumer's first–order condition is that her MRS equal the price ratio, therefore, she sets

$$\exp(x_1 - x_2) = \frac{p_2}{p_1} \tag{3 – 1}$$

where p_1 and p_2 are the prices of the two goods. Taking natural logarithms of both sides of equation $(3-1)$,

$$x_1 - x_2 = \ln p_1 - \ln p_2 \tag{3 – 2}$$

where I have used the fact that $\ln(a/b) = \ln a - \ln b$.

If we substitute for x_2 from the consumer's budget constraint,

$$x_2 = \frac{y - p_1 x_1}{p_2} \tag{3 – 3}$$

where y is the consumer's income, then equation $(3-2)$ becomes

$$x_1 - \frac{y}{p_2} + \frac{p_1}{p_2} x_1 = \ln p_1 - \ln p_2 \tag{3 – 4}$$

Or

$$x_1(1 + \frac{p_1}{p_2}) = \frac{y}{p_2} + \ln p_2 - \ln p_1 \tag{3 – 5}$$

so that the Marshallian demand function for good 1 is

$$x_1^M(p_1, p_2, y) = \frac{1}{p_1 + p_2} y + \frac{p_2}{p_1 + p_2} [\ln p_2 - \ln p_1] \tag{3 – 6}$$

and similarly, the Marshallian demand function for good 2 is

$$x_2^M(p_1, p_2, y) = \frac{1}{p_1 + p_2} y + \frac{p_1}{p_1 + p_2} [\ln p_1 - \ln p_2] \tag{3 – 7}$$

[Equations $(3-6)$ and $(3-7)$ make sense only if they imply non–negative values for consumption of each good. Otherwise, we have a corner solution. If

$$\ln p_1 - \ln p_2 > \frac{y}{p_2}$$

then we have a corner solution in which $x_1 = 0$ and $x_2 = y/p_2$, and if

$$\ln p_2 - \ln p_1 > \frac{y}{p_1}$$

then $x_1 = y/p_1$ and $x_2 = 0$.]

4. *Q* If a person's preferences can be represented by the direct utility function

$$u(x_1, x_2) = 100 - \frac{1}{x_1} - \frac{1}{x_2}$$

find the person's Marshallian demand functions for each good, her indirect utility function, her Hicksian demand function, and her expenditure function.

A Here the marginal utilities are

$$u_1 = [\frac{1}{x_1}]^2$$

$$u_2 = [\frac{1}{x_1}]^2$$

so that

$$MRS \equiv \frac{u_2}{u_1} = (\frac{x_1}{x_2})^2$$

and the consumer's first–order condition for utility maximization is

$$(\frac{x_1}{x_2})^2 = \frac{p_2}{p_1} \qquad (4-1)$$

which implies that

$$x_2 = x_1(\frac{\sqrt{p_1}}{\sqrt{p_2}}) \qquad (4-2)$$

Substituting from equation $(4-2)$ into the budget constraint,

$$p_1 x_1 + p_2(\frac{\sqrt{p_1}}{\sqrt{p_2}})x_1 = y \qquad (4-3)$$

or

$$x_1 = \frac{y}{\sqrt{p_1}(\sqrt{p_1} + \sqrt{p_2})} \qquad (4-4)$$

and similarly

$$x_2 = \frac{y}{\sqrt{p_2}(\sqrt{p_1} + \sqrt{p_2})} \qquad (4-5)$$

Equations $(4-4)$ and $(4-5)$ are the Marshallian demand functions for the two goods.

They also could be derived using equations $(E10)$ and $(E11)$ in *Jehle and Reny*, page 26 – since these preferences are examples of CES preferences. This utility function could be transformed into

$$U(x_1, x_2) = -([x_1]^{-1} + [x_2]^{-1})$$

by adding 100, which is a monotonic transformation. Then letting $V(x_1, x_2) = -[U(x_1, x_2)]^{-1}$ (a monotonic transformation) makes the utility function

$$V(x_1, x_2) = ([x_1]^{-1} + [x_2]^{-1})^{-1}$$

which is CES.

The indirect utility function, expenditure funaction, and Hicksian demand functions can all be calculated using the textbook's formulae for CES preferences.

But they also can be obtained directly. Substituting for x_1 and x_2 into the utility function from equations $(4-4)$ and $(4-5)$

$$v(p_1, p_2, y) = 100 - \frac{[\sqrt{p_1} + \sqrt{p_2}]\sqrt{p_1}}{y} - \frac{[\sqrt{p_1} + \sqrt{p_2}]\sqrt{p_2}}{y}$$

so that

$$v(p_1, p_2, y) = 100 - \frac{[\sqrt{p_1} + \sqrt{p_2}]^2}{y} \qquad (4-6)$$

To find the expenditure function, use the fact that $v(p_1, p_2, e(p_1, p_2, u) = u$, so that equation $(4-6)$ implies that

$$u = 100 - \frac{[\sqrt{p_1} + \sqrt{p_2}]^2}{e(p_1, p_2, u)}$$

meaning that

$$e(p_1, p_2, u) = \frac{[\sqrt{p_1} + \sqrt{p_2}]^2}{100 - u} \qquad (4-7)$$

The Hicksian demand functions are the derivatives of the expenditure function with respect to the prices, or

$$x_1^H(p_1, p_2, u) = e_1(p_1, p_2, u) = \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}(100 - u)} \qquad (4-8)$$

$$x_2^H(p_1, p_2, u) = e_1(p_1, p_2, u) = \frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_2}(100 - u)} \qquad (4-9)$$

These Hicksian demands can also be obtained by minimizing $p_1 x - 1 + p_2 x_2$ subject to the constraint $100 - 1/x_1 - 1/x_2 = u$.

5. Q A person's preferences are described as **quasi–linear** if they can be represented by the utility function

$$u(x_1, x_2, \ldots, x_n) = x_1 + g(x_2, x_3, \ldots, x_n)$$

for some increasing, concave function $g : R^{n-1} \rightarrow R$.

If a person's preferences are quasi–linear, what is the income elasticity of demand for each good?

Explain briefly.

A Given these quasi–linear preferences, the person's first–order conditions for utility maximization are

$$u_1 = 1 = \lambda p_1 \tag{5 – 1}$$

$$u_i = g_i = \lambda p_i \quad i = 2, 3, \cdots, n \tag{5 – 2}$$

where λ is the Lagrange multiplier on the budget constraint in the consumer's utility maximization problem.

That means that the first–order conditions for consumption of goods $2, 3, \ldots, n$ can be written

$$u_i = \frac{1}{p_1} g_i(x_2, x_3, \ldots, x_n) \quad i = 2, 3, \cdots, n \tag{5 – 3}$$

Equation $(5 - 3)$ defines the $n - 1$ levels of consumption $x_2, x_3, \ldots, x_n$ of all goods but #1, as functions of the n prices $p_1, p_2, p_3, \ldots, p_n$. So the system $(5 - 3)$ can be solved for the Marshallian demand functions for goods # $2, 3, \ldots, n$. But the level y of income does not appear in equation $(5 - 3)$; changing y will not affect the levels of consumption $x_2, x_3, \ldots, x_n$ which solve equation $(5 - 3)$, provided that the prices $p_1, p_2, p_3, \ldots, p_n$ are unchanged.

That is, the income elasticity of demand for all goods except good #1 is zero.

Since the budget constraint is

$$y = p_1 x_1 + \cdots + p_n x_n \tag{5 – 4}$$

differentiation of equation $(5 - 4)$ with respect to y implies that

$$1 = p_1 \frac{\partial x_1}{\partial y} \tag{5 – 5}$$

when account is taken of the fact that

$$\frac{\partial x_i}{\partial y} = 0 \quad i = 2, 3, \cdots, n$$

So the income elasticity of demand for good #1 is

$$\eta_y^1 = \frac{\partial x_1}{\partial y} \frac{y}{x_1} = \frac{y}{p_1 x_1} = \frac{1}{s_1}$$

where s_i is the share of the person's income which she chooses to spend on good $\#i$.