*Q*1. The table below indicates the prices $\mathbf{p}^t$ of three commodities, at 3 different times t, and the consumption bundle $\mathbf{x}^t$ actually chosen by the consumer at each of the three times.

What can be said about the consumer's preferences over the 3 bundles $\mathbf{x}^t$?

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	3	2	1	4	6	12
2	2	5	1	5	2	10
3	8	5	4	5	8	10

*A*1 The table below shows the cost of bundle i in year j. For example, the number 29 in the second column in row 1 indicates that consumption bundle $\mathbf{x}^2$, the bundle actually chosen in year 2, would have cost 29 in year 1, using year 1 prices (p_1^1, p_2^1, p_3^1).

36	29	41
50	30	60
110	90	120

In row 1, the cost in the second column is less than the cost in the first. That is,$\mathbf{x}^2$ cost less than $\mathbf{x}^1$ in year 1. So $\mathbf{x}^1$ has been revealed preferred to $\mathbf{x}^2$: the consumer actually chose $\mathbf{x}^1$ when she could have afforded $\mathbf{x}^2$.

Whenever the number in row i, column j, is less than or equal to the number on the diagonal (row i, column i), then $\mathbf{x}^i$ has been revealed preferred to $\mathbf{x}^j$.

So the first row in the table above shows that $\mathbf{x}^1$ is revealed preferred to $\mathbf{x}^2$; the second row shows nothing, the third row shows that $\mathbf{x}^3$ is revealed preferred to both $\mathbf{x}^1$ and $\mathbf{x}^2$.

Thus none of the data in this question indicate any violation of the strong (or weak) axiom of revealed preference. All the observations are consistent with the consumer ranking $\mathbf{x}^3$ highest, followed by $\mathbf{x}^1$, followed by $\mathbf{x}^2$.

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	5	10	10	10
2	4	2	1	5	20	9
3	3	3	3	7	12	15
4	3	1	2	8	15	12

*A*2. Here the matrix of costs of the bundles is

70	70	94	83
70	69	67	74
90	102	102	105
60	53	63	63

The first row shows that $\mathbf{x}^1$ is revealed preferred to $\mathbf{x}^2$. The second row shows that $\mathbf{x}^2$ is revealed preferred to $\mathbf{x}^3$. The third row shows that $\mathbf{x}^3$ is revealed preferred to $\mathbf{x}^1$, and to $\mathbf{x}^2$. The fourth row shows that $\mathbf{x}^4$ is revealed preferred to $\mathbf{x}^1$, to $\mathbf{x}^2$, and to $\mathbf{x}^3$.

So there is one violation of $WARP$ (as defined on page 87 of the text). Row 3 shows that $\mathbf{p}^3 \cdot \mathbf{x}^2 \leq \mathbf{p}^3 \cdot \mathbf{x}^3$, so that $WARP$ will hold only if $\mathbf{p}^2 \cdot \mathbf{x}^3 > \mathbf{p}^2 \cdot \mathbf{x}^2$, which row 2 shows is not the case.

In addition, there is another violation of $SARP$, which is not a violation of $WARP$: row 1 shows that $\mathbf{x}^1$ is revealed preferred to $\mathbf{x}^2$; row 2 shows that $\mathbf{x}^2$ is revealed preferred to $\mathbf{x}^3$; row 3 shows that $\mathbf{x}^3$ is revealed preferred to $\mathbf{x}^1$.

*Q*3. *i* If a person's utility–of–wealth function has the equation $u(W) = A - e^{-\alpha W}$, where A and α are positive parameters, what is her coefficient of absolute risk aversion?

ii If a person could invest her wealth in a safe asset, offering a certain rate of return $r \geq 0$, or a risky asset, which offers the return $r_g > r$ with some probability π, and the return $r_b < r$ with probability $1 - \pi$, how much wealth should she invest in the safe asset, and how much in the risky asset, if her utility–of–wealth function is $U(W) = A - e^{-\alpha W}$?

*A*3. *i* In this case,

$$u'(W) = \alpha e^{-\alpha W}$$

$$u''(W) = -(\alpha)^2 e^{-\alpha W}$$

since the derivative of $e^{f(x)}$ with respect to x is $f'(x)e^{f(x)}$.

So that means that the coefficient R_A of absolute risk aversion is just the positive constant α.

ii If the person invested an amount X in the risky asset, and the remainder of her wealth in the safe asset, then her expected utility would be

$$A - \pi e^{-\alpha[(1+r_g)X+(W-X)(1+r_s)]} - (1-\pi)e^{-\alpha[(1+r_b)X+(1+r_s)(W-x)]} \qquad (3-1)$$

The derivative of expression $(3-1)$ with respect to X is

$$\alpha[\pi(r_g - r_s)e^{-\alpha[(1+r_g)X+(W-X)(1+r_s)]} + (1-\pi)(r_b - r_s)e^{-\alpha[(1+r_b)X+(1+r_s)(W-x)]} \qquad (3-2)$$

To maximize her expected utility, the person should choose an investment X in the risky asset which makes the derivative of expected utility with respect to X, expression $(3-2)$, equal 0. Setting this expression equal to zero means setting

$$\frac{\pi}{(1-\pi)}\frac{r_g - r_s}{r_s - r_b} = \frac{e^{-\alpha[(1+r_b)X+(1+r_s)(W-x)]}}{e^{-\alpha[(1+r_g)X+(W-X)(1+r_s)]}} \qquad (3-3)$$

Using the fact that

$$\frac{e^a}{e^b} = e^{a-b}$$

equation $(3-3)$ can be written

$$\frac{\pi}{(1-\pi)}\frac{r_g - r_s}{r_s - r_b} = e^{\alpha(r_g - r_b)X} \qquad (3-4)$$

Taking natural logarithms of both sides of $(3-4)$, and using the fact that $\ln ab = \ln a + \ln b$,

$$X = \frac{\ln \pi - \ln(1-\pi) + \ln r_g - r_s - \ln(r_s - r_b)}{\alpha(r_g - r_s)} \qquad (3-5)$$

Since the person has $CARA$ preferences, the amount of money she wishes to invest in the risky asset does not vary with her wealth W.

*Q*4. What is the risk premium for an investment which yields a prize of G, with probability $1/G$, (and nothing with probability $(G-1)/G$), to a person with the utility–of–wealth function

$$u(W) = A - \frac{b}{W}$$

where $b > 0$?

*A*4. Her expected utility if she undertakes the investment is

$$EU \equiv A - \frac{G-1}{G}\frac{b}{W} - \frac{1}{G}\frac{b}{W+G} \qquad (4-1)$$

The certainty equivalent CE to the investment is the certain amount of money CE such that $u(W+CE) = EU$, or

$$A - \frac{b}{W+CE} = A - \frac{G-1}{G}\frac{b}{W} - \frac{1}{G}\frac{b}{W+G} \qquad (4-2)$$

which is equivalent to

$$\frac{G}{W+CE} = \frac{G-1}{W} + \frac{1}{W+G} \qquad (4-3)$$

or

$$GW(W+G) = (G-1)(W+CE)(W+G) + W(W+CE) \qquad (4-4)$$

or

$$CE[(G-1)(W+G)+W] = W[G(W+G)-(G-1)(W+G)-W] \qquad (4-5)$$

which can be simplified to

$$CE = \frac{W}{W+G-1} \qquad (4-6)$$

The risk premium is simply the difference between the expected value of the investment, and the certainty equivalent. The expected value of the investment here is just 1 — winning G with probability $1/G$. So

$$P = 1 - \frac{W}{W+G-1} = \frac{G-1}{W+G-1} \qquad (4-7)$$

The risk premium is 0 if $G = 1$, since then the investment carries no risk (win \$1 for sure). It increases with G, since increasing G imposes a mean–preserving spread on the distribution of the returns to the investment. And it decreases with the person's wealth level W, since the utility–of–wealth function $u(W) = A - b/W$ exhibits constant relative risk aversion, and therefore also exhibits decreasing absolute risk aversion.

*Q*5. If a production function $f(x_1, x_2)$ has the equation

$$f(x_1, x_2) = [a + b\frac{x_1}{x_2}]^{-1}x_1$$

for positive parameters a, and b, calculate the marginal product of each input, and the marginal rate of technical substitution. Does the production function exhibit decreasing, constant, or increasing returns to scale? Explain briefly.

*A*5. The production function can also be written

$$f(x_1, x_2) = \frac{x_1 x_2}{a x_2 + b x_1}$$

so that

$$f_1(x_1, x_2) = \frac{a(x_2)^2}{(a x_2 + b x_1)^2}$$

$$f_2(x_1, x_2) = \frac{b(x_1)^2}{(a x_2 + b x_1)^2}$$

and the $MRTS$ is

$$MRTS(x_1, x_2) = \frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \frac{a}{b}(\frac{x_2}{x_1})^2$$

Moving down and to the right along an isoquant — increasing x_1 and lowering x_2 — lowers the $MRTS$, so that the function exhibits a decreasing $MRTS$.

To examine returns to scale, the easiest method is probably to compute directly $f(tx_1, tx_2)$ for some scalar $t > 0$.

$$f(tx_1, tx_2) = \frac{(tx_1)(tx_2)}{atx_2 + btx_1}$$

which means that

$$f(tx_1, tx_2) = \frac{t^2(x_1 x_2)}{t(ax_2 + bx_1)} = tf(x_1, x_2)$$

so that the function exhibits constant returns to scale.

In fact, whenever a production function $f(\mathbf{x})$ can be written

$$F(\mathbf{x}) = g(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \cdots, \frac{x_n}{x_1})x_1$$

for some function $g(\cdot)$, then it will exhibit constant reurns to scale.