GS/ECON 5010 Answers to Assignment 3 W2005

*Q*1. What are the market price, and aggregate quantity sold, in long–run equilibrium in a perfectly competitive market for which the demand function has the equation

$$Q = 2000 - 5p$$

(where Q is aggregate quantity demanded, and p the price), if there is free entry by identical firms to the industry, each of which has the long–run total cost function

$$TC = 600q - 80q^2 + 4q^3$$

where q is the quantity produced by the firm?

*A*1. Given the total cost function, the average cost function is

$$AC = 600 - 80q + 4q^2 \qquad (1-1)$$

and the marginal cost function is

$$MC = 600 - 160q - 12q^2 \qquad (1-2)$$

From $(1-1)$,

$$AC' = 8q - 80 \qquad (1-3)$$

which shows that the average cost curve is U–shaped : it decreases with q if and only if $8q < 80$, or $q < 10$.

At $q = 10$,

$$AC = 600 - 80(10) + 4(10)^2 = 200$$

and

$$MC = 600 - 160(10) + 12(10)^2 = 200$$

so that the marginal cost curve cuts the average cost curve at the minimum average cost, as required.

In long–run equilibrium, with free entry, each firm must be operating at its minimum average cost, which here is $AC = 200$, attained at an output level of $q = 10$.

Due to free entry (and the U–shaped average cost curve), the long–run supply curve is horizontal, at a height of 200. The equilibrium price must be 200, no matter what the demand curve.

So $p = 200$, which means, from the equation of the demand curve, that aggregate quantity demanded Q is $2000 - 5(200) = 1000$. In long–run equilibrium, there will be 100 identical firms in the industry, each producing 10 units at a cost of 200 per unit.

$Q2$. Suppose that a market contains 1 million identical consumers, each of whom has preferences which can be represented by the utility function

$$U(X, q_1, q_2) = X + (q_1^\alpha + q_2^\alpha)^\beta$$

where X is consumption of a numéraire good, and q_1 and q_2 are consumption of goods produced by firms #1 and #2 respectively, and where $\alpha < 1$ and $\beta < 1$.

Firms 1 and 2 each have the identical total cost function, $C(q, \mathbf{w}) = cq$, where c is a constant.

i Which levels of output for the two firms would maximize their combined profits?

ii What levels of output would firms #1 and #2 produce if they behaved as Cournot duopolists?

(You may assume that $q_1 = q_2$ in the solutions to each of the problems.)

$A2$. Given the preferences of consumers, what is each consumer's inverse demand function? The first–order conditions for consumer utility maximization imply that

$$U_X = 1 = \lambda p_X = \lambda \tag{2-1}$$

$$U_1 = \alpha\beta(q_1^\alpha + q_2^\alpha)^{\beta-1} q_1^{\alpha-1} = \lambda p_1 \tag{2-2}$$

$$U_2 = \alpha\beta(q_1^\alpha + q_2^\alpha)^{\beta-1} q_2^{\alpha-1} = \lambda p_2 \tag{2-3}$$

where U_i is the marginal utility with respect to q_i.

Since good X is numéraire, equation $(2-1)$ implies that $\lambda = 1$, so that equations $(2-2)$ and $(2-3)$ define the inverse demand functions of the two firms. If firm 1 produces q_1 million units of output, and firm 2 produces q_2 million, then equations $(2-2)$ and $(2-3)$ define the prices p_1 and p_2 that will clear the markets for the two goods.

So if firm 1 produces q_1 million units, its profits will be $(p_1 - c)q_1$, where

$$p_1 = \alpha\beta(q_1^\alpha + q_2^\alpha)^{\beta-1} q_1^{\alpha-1}$$

and similarly, firm 2 will earn profits of $(p_2 - c)q_2$, where

$$p_2 = \alpha\beta(q_1^\alpha + q_2^\alpha)^{\beta-1} q_2^{\alpha-1}$$

i Joint profits of the two firms are $(p_1 - c)q_1 + (p_2 - c)q_2$. If $q_1 = q_2 = q$, then

$$p_1 = p_2 = \alpha\beta[2q^\alpha]^{\beta-1} q^{\alpha-1} = \alpha\beta 2^{\beta-1} q^{\alpha\beta-1}$$

so that joint profits are

$$2(\alpha\beta 2^{\beta-1} q^{\alpha\beta-1} - c)q = 2[(\alpha\beta 2^{\alpha(\beta-1)} q^{\alpha\beta}) - cq] \tag{2-4}$$

Maximizing expression $(2-4)$ with respect to q implies that

$$\alpha^2\beta^2 2^{\beta-1} q^{\alpha\beta-1} = c \qquad (2-5)$$

or

$$q^* = c^\gamma[\alpha^2\beta^2 2^{\beta-1}]^{-\gamma} \qquad (2-6)$$

where

$$\gamma = \frac{1}{\alpha\beta - 1}$$

ii If firm 1 behaves as a Cournot duopolist, it chooses its own output q_1 to maximize its own profit $(p_1 - c)q_1$, taking the other firm's output q_2 as **given**. That means it tries to maximize

$$\alpha\beta(q_1^\alpha + q_2^\alpha)^{\beta-1} q_1^\alpha - cq_1$$

with respect to q_1.

The first–order condition for firm 1's profit maximization is

$$\alpha^2\beta[(q_1^\alpha + q_2^\alpha)^{\beta-2} q_1^{\alpha-1}][(\beta-1)q_1^\alpha + (q_1^\alpha + q_2^\alpha)] - c = 0 \qquad (2-7)$$

Firm 2 has an analogous optimality condition

$$\alpha^2\beta[(q_1^\alpha + q_2^\alpha)^{\beta-2} q_2^{\alpha-1}][(\beta-1)q_2^\alpha + (q_1^\alpha + q_2^\alpha)] - c = 0 \qquad (2-8)$$

In Cournot–Nash equilibrium, both firms are on their reaction functions, defined by equations $(2-7)$ and $(2-8)$. If the equilibrium is symmetric, so that $q_1 = q_2 = q$, equation $(2-7)$ (or $(2-8)$) becomes

$$\alpha^2\beta[2q^\alpha 2^{\beta-2} q^{\alpha-1}[(\beta-1)q^\alpha + 2q^\alpha] = c \qquad (2-9)$$

or

$$\alpha^2\beta(\beta+1)2^{\beta-2} q^{\alpha\beta-1} = c \qquad (2-10)$$

so that

$$q^c = c^\gamma[\alpha^2\beta(\beta+1)2^{\beta-2}]^{-\gamma} \qquad (2-11)$$

Comparing $(2-6)$ and $(2-10)$,

$$q^c = [\frac{2\beta}{\beta+1}]^\gamma q^*$$

Since $\beta < 1$, $2\beta = \beta + \beta < \beta + 1$. But since $\gamma < 0$, therefore, $q^c > q^*$.

*Q*3. Solve for the quantity produced by each firm, the price, and each firm's profits, if there were J firms acting as Cournot oligopolists, each producing a homogeneous good, for which the market demand is linear

$$p = a - bQ$$

where $Q \equiv q_1 + q_2 + \cdots + q_J$ was industry output, if each firm had the (same) total cost function

$$C(q) = cq^2$$

for some positive constant c?

*A*4 Each firm's profits $pq_i - c(q_i)^2$ would be

$$\pi_i = (a - b[q_1 + q_2 + \cdots + q_J])q_i - c(q_i)^2 \qquad (3-1)$$

maximizing π_i with respect to q_i yields the first–order condition

$$a - bQ_{\_i} = 2(b + c)q_i \qquad (3-2)$$

where $Q_{\_i} \equiv \sum_{j \neq i} q_j$ is the sum of all other firms' quantities produced.

In a symmetric Nash equilibrium, $q_1 = q_2 = \cdots = q_J = q$, so that $Q_{\_i} = (J-1)q$ and equation $(3-2)$ becomes

$$[(J+1)b + 2c]q = a \qquad (3-3)$$

or

$$q = \frac{a}{(J+1)b + 2c} \qquad (3-4)$$

Industry output Q equals Jq in a symmetric Nash equilibrium, so that

$$Q = \frac{aJ}{(J+1)b + 2c} \qquad (3-5)$$

Since $p = a - bQ$,

$$p = a\frac{b + 2c}{(J+1)b + 2c}$$

and the profit of each firm is

$$\pi_i = p_i q_i - cq_i^2 = \frac{a^2(b+2c)}{[(J+1)b+2c]^2} - \frac{a^2c}{[(J+1)b+2c]^2} = [\frac{a^2(b+c)}{[(J+1)b+2c]^2}]^2$$

So (as in the case of constant marginal costs), total industry profit decreases with the number of firms J in the industry.

*Q*4. What would be the equilibrium price, and the equilibrium profits of each firm, in a market with two Bertrand oligopolists, producing goods which are imperfect substitutes for each other, with quantity demanded of the products of the two firms being

$$q_1 = \frac{p_1^{r-1}}{p_1^r + p_2^r}$$

$$q_2 = \frac{p_2^{r-1}}{p_1^r + p_2^r}$$

where $r < 0$, if each firm's cost of producing c units were cq (where c is a positive constant).. (You may restrict attention to symmetric equilibria, in which $p_1 = p_2$.)

*A*4. The profit of firm 1, if it charges a price p_1 (and if firm 2 charges a price of p_2), is

$$(p_1 - c)q_1 = \frac{p_1^r}{p_1^r + p_2^r} - \frac{cp_1^{r-1}}{p_1^r + p_2^r} \qquad (4-1)$$

Maximizing this profit with respect to p_1 gives the first–order condition

$$\frac{p_1^{r-1}}{[p_1^r + p_2^r]^2}[rp_2^r - c(r-1)p_1^{r-1} - c(r-1)p_2^r p_1^{-1} + crp_1^{r-1}] = 0 \qquad (4-2)$$

In a symmetric equilibrium, in which $p_1 = p_2 = p$, equation $(4-2)$ becomes

$$rp = c(r-2)$$

or

$$p = \frac{r-2}{r}c \qquad (4-3)$$

*Q*5. Another model of duopoly is that of **von Stackelberg**, in which firms choose output levels **sequentially**. That is, firm 1 chooses its output. Firm 2 observes what output level fiirm 1 has chosen, and then chooses its own output level. What output levels would the 2 firms choose, if they behaved in this manner, if they both produced an identical product for which the market inverse demand function had the equation

$$p = 21 - (q_1 + q_2)$$

if each firm had a total cost function

$$TC = \begin{matrix} 4 + q_i & \text{if} & q_i > 0 \\ 0 & \text{if} & q_i = 0 \end{matrix}$$

where q_i is the output level of firm i?

*A*5. This problem must be solved backwards. First, what is firm 2's reaction to firm 2 producing an output level of q_1? If $q_2 > 0$, then

$$\pi_2 = pq_2 - TC(q_2) = [21 - (q_1 + q_2)]q_2 - (4 + q_2) \qquad (5-1)$$

Maximizing π_2 with respect to q_2 yields the first–order condition

$$21 - q_1 - 2q_2 - 1 = 0 \qquad (5-2)$$

or

$$q_2 = 10 - \frac{q_1}{2} \tag{5 – 3}$$

But firm 2 will choose to produce a positive level of output only if it earns a positive profit. What is its profit if firm 1 has chosen an output level of q_1, and if firm 2 has responded by choosing $q_2 = 10 - (q_1/2)$? In this case the price is $21 - q_1 - q_2$, which means that

$$p = 21 - q_1 - (10 - \frac{q_1}{2}) = 11 - \frac{q_1}{2} \tag{5 – 4}$$

Substituting back into $(5 - 1)$,

$$\pi_2 = [10 - \frac{q_1}{2}][p - 1] - 4 = [10 - \frac{q_1}{2}]^2 - 4 \tag{5 – 5}$$

So firm 2 can earn a positive profit only if

$$[10 - \frac{q_1}{2}]^2 > 4$$

which is the same thing as

$$10 - \frac{q_1}{2} > 2$$

or

$$q_1 < 16$$

So if $q_1 \geq 16$, then firm 2's best response is to produce nothing at all, since the fixed costs (of 4) imply that it would lose money at any positive level of production. If $q_1 < 16$, firm 2 should produce the output level defined by equation $(5 - 3)$.

Now consider firm 1's decision. It knows that if it produces an output level of $q_1 < 16$, then firm 2 will follow by producing $10 - \frac{q_1}{2}$, resulting in a price of $11 - \frac{q_1}{2}$. So firm 1's profit, if it chooses an output level of q_1 initially, will be

$$\pi_1 = pq_1 - TC(q_1) = [10 - \frac{q_1}{2}]q_1 - 4 \tag{5 – 6}$$

Maximizining $(5 - 6)$ with respect to q_1 yields the first–order condition

$$q_1 = 10$$

resulting in profits of

$$\pi_1 = [10 - \frac{10}{2}]10 - 4 = 46$$

On the other hand, if firm 1 produces an output of 16 or more, then firm 2 will shut down completely. That would result in a price of $21 - q - 1$, and a profit to firm 1 of

$$(21 - q_1)q_1 - q_1 - 4 = 20q_1 - q_1^2 - 4 \tag{5 – 7}$$

The expression $(5 - 7)$ is decreasing in q_1 when $q_1 \geq 16$. That means that, if firm 1 were to find it profitable to have $q_1 \geq 16$, that $q_1 = 16$ would be the best level of output to choose. That level is the smallest level of output for firm 1 which will induce firm 2 to shut down completely.

At $q_1 = 16$ (and $q_2 = 0$), equation $(5 - 7)$ shows that

$$\pi_1 = 20(16) - 256 - 4 = 60$$

Since $60 > 46$, then the best strategy for firm 1 is to produce an output just high enough that firm 2 cannot make a profit. The Stackelberg equilibrium here has $q_1 = 16$ and $q_2 = 0$.