*Q*1. What does the contract curve look like for a 2–person, 2–good exchange economy, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = 100 - \frac{1}{x_1^1} - \frac{1}{x_2^1}$$

$$u^2(x_1^2, x_2^2) = x_2^2 + 100 \ln x_1^2$$

where x_j^i is person i's consumption of good j?

*A*1. An allocation inside the Edgeworth box is on the contract curve if the two people have the same MRS's.

For these two people,

$$MRS_1 = \frac{u_2^1}{u_1^1} = [\frac{x_1^1}{x_2^2}]^2 \tag{1-1}$$

$$MRS_2 = \frac{u_2^2}{u_1^2} = \frac{x_1^2}{100} \tag{1-2}$$

If the total endowments of the two goods in the economy are X_1 and X_2 respectively, then $x_1^2 = X_1 - x_1^1$, so that $MRS_1 = MRS_2$ if and only if

$$[\frac{x_1^1}{x_2^2}]^2 = \frac{X_1 - x_1^1}{100} \tag{1-3}$$

Equation $(1-3)$ defines a curve

$$x_2^1 = \frac{10x_1^1}{\sqrt{X_1 - x_1^1}} \tag{1-4}$$

The curve defined by $(1-4)$ is (part of) the contract curve. The curve starts at the bottom–left corner ($x_1^1 = 0, x_2^1 = 0$) of the Edgeworth box. It slopes up. From equation $(1-4)$, the second derivative of x_2^1 with respect to x_1^1 is also positive : so the curve slopes up at an increasing rate.

The curve also does not hit the right side of the Edgeworth box : equation $(1-4)$ shows that x_2^1 would approach ∞ as $x_1^1 \rightarrow X_1$. So the curve hits the top of the Edgeworth box, at the point at which

$$\frac{10x_1^1}{\sqrt{X_1 - x_1^1}} = X_2 \tag{1-5}$$

which happens to be the value x^* for which

$$x^* = \frac{\sqrt{(X_2)^4 + 400X_1(X_2)^2}}{200} - \frac{(X_2)^2}{200} \tag{1-6}$$

The contract curve then follows the top of the Edgeworth box, from (x^*, X_2) to (X_1, X_2) : the allocations which are most favourable to person 1 have person 2 consuming none of good 2, since MRS_2 approaches 0 as $x_1^2 \rightarrow 0$.

*Q*2. What are all the allocations in the core of a 4–person exchange economy in which all 4 people had the same preferences, represented by the utility function

$$u^i(x_1^i, x_2^i) = x_1^i x_2^i$$

if person 1 and person 2 each had the endowment vector $(2, 0)$, and if person 3 and person 4 each had the endowment vector $(0, 2)$?

(i) If an allocation is in the core, then it must be Pareto optimal. (Otherwise, it could be blocked by a coalition of all 4 people.)

An allocation is Pareto optimal if everyone's MRS's are the same. Here, each person's MRS is x_1^i / x_2^i. So, in order for an allocation to be Pareto optimal here, each person must consume the two goods in the same proportions.

Since the total endowment of each good here is 4, then each person can consume the goods in the same proportion only if $x_1^i = x_2^i$, for each person i.

(ii) Any allocation in the core must give the exact same allocation to person 1 as to person 2. Why? Suppose that some allocation gave person 1 (a, a), person 2 (b, b), person 3 (c, c) and person 4 (d, d), with $a > b$. (From part (i), we know that each core allocation must give the same quantity of each good to each person.) Suppose that $c \geq d$. Then we must have $a + c > 2 > b + d$. If person 2 and person 4 formed a two–person coalition, then they would have more than 2 units of each good to allocate, so that this coalition could give person 2 a little more than b of each good, and person 4 a little more than d, blocking the proposed allocation.

This blocking would occur whenever person 1 and person 2 were treated differently, or whenever person 3 and person 4 are treated differently : a coalition of the worst–treated of $\{1, 2\}$ and the worst–treated of $\{3, 4\}$ could block the allocation.

(iii) So now the core allocations have been shown to be allocations of the type $(a, a), (a, a), (c, c), (c, c)$, with $a + c = 2$.

Suppose that $a < c$. How low can a be, without being blocked? What could persons 1 and 2 do, if they were given only $a < c$ of each good?

They could try and form a coalition with person 3, leaving out person 4. To get person 3 to join the coalition, they must offer her at least the utility she would get from the proposed allocation, which is c^2. The proposed coalition would have 4 units of good 1, and 2 units of good 2 in total. So the coalition's problem is to give y units of good 1 to person 3, and z units of good 2 to person 3, so as to give person 3 a utility of at least c^2, and to maximize the utility that person 1 and 2 get from what is left. Person 1 and person 2 would have $(4 - y)/2$ units of good 1 each,and $(2 - z)/2$ units of good 2. So their problem is to maximize

$$[\frac{4-y}{2}][\frac{2-z}{2}]$$

subject to

$$yz \geq c^2$$

Setting up the Lagrangian,

$$\mathcal{L} = [\frac{4-y}{2}][\frac{2-z}{2}] + \lambda(yz - c^2) \tag{2-1}$$

The first–order conditions for optimality for this Lagrangian are

$$\frac{2-z}{4} = \lambda z \tag{2-2}$$

$$\frac{4-y}{4} = \lambda y \tag{2-3}$$

which imply

$$y = 2z \tag{2-4}$$

The rationale for equation $(2-4)$? The coalition, if it wants to give the highest utility to its members, should divide up the allocation's endowment efficiently, so that each coalition member has the same MRS. (Otherwise people 1 and 2 could make themselves better off, while still offering utility of c^2 to person 3.)

So person 3 must get a bundle $(2z, z)$, with $2z^2 \leq c^2$, if she is to be induced into joining the blocking coalition. Person 1 and 2 will try to give the minimum possible level of utility to person 3 (subject to getting her to join), so that they will choose z such that $2z^2 = c^2$ or

$$z = \frac{c}{\sqrt{2}} \tag{2-5}$$

What does that leave person 1 and person 2? If person 3 gets $(2z, z)$, then they each get $(2-z, \frac{2-z}{2})$, giving them a utility level of

$$\frac{(2-z)^2}{2}$$

From equation $(2-5)$, the best that person 1 and person 2 can get from the blocking coalition is

$$\frac{(2-\frac{c}{\sqrt{2}})^2}{2}$$

What did they get from the original allocation? Since $a + c = 2$, they each got an allocation of $(2-c, 2-c)$, and utility of $(2-c)^2$. Their proposed 3–person coalition will give them higher utility if and only if

$$\frac{(2-\frac{c}{\sqrt{2}})^2}{2} \geq (2-c)^2 \tag{2-6}$$

which is the same condition as

$$2 - \frac{c}{\sqrt{2}} \geq \sqrt{2}(2-c) \tag{2-7}$$

Condition $(2-7)$ is

$$c \geq 2(2 - 2\sqrt{2}) \tag{2-8}$$

which means that they can successfully block any allocation in which $c \geq 1.18$, approximately, since

$$\sqrt{2} \approx 1.41$$

So if $c > 1.18$ (that is, if $a < 0.82$), person 1 and 2 can organize a blocking coalition of $\{1, 2, 3\}$, which can give all three members higher utility than the proposed allocation.

Similarly, if $c < 0.82$, then person 3 and 4 could organize a blocking coalition $\{1, 3, 4\}$.

(iv) So what allocations cannot be blocked by any coalition? Combining parts (i), (ii) and (iii), they are allocations of the form

$$(a, a), (a, a), (2 - a, 2 - a), (2 - a, 2 - a)$$

for which

$$2(2 - 2\sqrt{2}) \geq a \geq 2(\sqrt{2} - 1)$$

*Q*3. How would the equilibrium prices of the goods vary with the people's endowments in a 2–person, 2–good exchange economy, if each person's preferences could be represented by the utility function

$$u^i((\mathbf{x}^i) = a \ln x_1^i + b \ln x_2^i$$

where x_j^i was person i's consumption of good j?

*A*3. To find the equilibrium, first the excess demand functions for the economy must be found. They follow from people's Marshallian demands.

Here, a person's MRS is

$$\frac{u_2^i}{u_1^i} = \frac{bx_1^i}{ax_2^i} \qquad (3-1)$$

To maximize utility,the person equates this MRS to the price ratio,so that

$$\frac{bx_1^i}{ax_2^i} = \frac{p_2}{p_1} \qquad (3-2)$$

implying that

$$x_1^i = \frac{a}{b}\frac{p_2}{p_1}x_2^i \qquad (3-3)$$

Substituting $(3-3)$ in a person's budget equation $p_1x_1^i + p_2x_2^i = y^i$,

$$p_1\frac{a}{b}\frac{p_2}{p_1}x_2^i + p_2x_2^i = y^i \qquad (3-4)$$

yielding the person's Marshallian demand functions

$$x_1^i(p_1, p_2, y^i) = \frac{a}{a+b}\frac{y^i}{p_1} \qquad (3-5)$$

$$x_2^i(p_1, p_2, y^i) = \frac{b}{a+b}\frac{y^i}{p_2} \qquad (3-6)$$

Here, the peron's income is the value of her endowment, so that, for example, her demand for good 1 is

$$\frac{a}{a+b}\frac{p_1 e_1^i + p_2 e_2^i}{p_1} = \frac{a}{a+b}[e_1^i + \frac{p_2}{p_1}e_2^i] \qquad (3-7)$$

Now a person's **excess demand** for a good is her demand, minus her endowment.

Since only relative prices matter, we can set

$$p \equiv \frac{p_2}{p_1}$$

and person i's excess demand for good 1 is

$$z(p, e_1^i, e_2^i) = \frac{1}{a+b}[ape_2^i - be_1^i] \qquad (3-8)$$

The aggregate excess demand for good 1 in the economy is the sum of each person's excess demand, so that here

$$Z(p, \mathbf{e}) = \frac{1}{a+b}[apE_2 - bE_1] \qquad (3-9)$$

where E_i is the economy's aggregate endowment of good i.

If the market for good 1 is to clear, the right side of equaion $(3-9)$ must equal zero, so that for a Walrasian equilibrium it is necessary that

$$p = \frac{b}{a}\frac{E_1}{E_2} \qquad (3-10)$$

But Walras's Law implies that the market for good 2 must clear if the market for good 1 clears. So any price vector (p_1, p_2) for which

$$\frac{p_2}{p_1} = \frac{b}{a}\frac{E_1}{E_2}$$

will be an equilibrium price vector for this economy.

*Q*4. Find all the Nash equilibria (pure and mixed) in the following strategic–form two–person game.

	LL	L	R	RR
tt	$(20,0)$	$(5,4)$	$(100,2)$	$(10,30)$
t	$(0,5)$	$(10,10)$	$(40,5)$	$(20,6)$
b	$(3,60)$	$(5,10)$	$(10,20)$	$(7,50)$
bb	$(4,40)$	$(8,50)$	$(20,60)$	$(12,60)$

*A*4 This game can be solved by iterated elimination of strictly dominated strategies. Row b is strictly dominated for player 1 by row bb, so that we can cross out row b. Once row b is crossed out, column LL is strictly dominated (by column L) for player 2. With column LL crossed out, player 1 finds bb strictly dominated (by row t). Now, with b and bb crossed out, column R is strictly dominated (by column L or RR) for player 2. That makes tt strictly dominated (by t) for player 1.

So iterated elimination of strictly dominated strategies leads to the solution (t, L) as a pure strategy Nash equilibrium.

That is the only Nash equilibrium. In general, if a game can be solved by iterated elimination of strictly dominated stratgies, then this solution is the only Nash equilibrium, in pure or mixed startegies.

Why? Player 1 would never put any probability on row b, since row bb gives her a higher expected payoff, whatever mixed strategy is being played by player 2. But if player 1 must put 0 probability on strategy b, then player 2 should never put any positive weight on strategy LL : if he knows that b will be played with probability 0, then L must give a higher expected payoff for 2 against any mixed strategy (in which t has zero weight) by player 1. But then player 1 realizes that t gives a higher expected payoff than bb, against any mixed strategy used by player 2 in which LL is given zero weight. And so on : the same logic which successively eliminated pure strategies can be used to show successively that no posiitve probability weight can be put on any of the eliminated strategies.

*Q*5. Find all the Nash equilibria (in pure or mixed strategies) to the following two–person game in strategic form.

	L	R
t	$(2, 6)$	$(6, 4)$
b	$(0, 4)$	$(10, 8)$

*A*5. There are two Nash equilibria in pure strategies (t, L) and (b, R).

To check for mixed stratey equilibria, see what mixed strategy by player 2 would induce player 1 to play a mixed strategy herself. Player 1 will be willing to play a mixed strategy only if she is indifferent between her pure strategies t and b. She will be indifferent only if they yield her the same expected payoff, that is if

$$2\tau + 6(1-\tau) = 10(1-\tau) \qquad (5-1)$$

where τ is the probability with which player 2 plays the pure strategy L. Equation $(5-1)$ implies that

$$\tau = \frac{2}{3}$$

Player 2 will be willing to randomize only if both his pure strategies give him the same expected payoff, that is if

$$6\sigma + 4(1-\sigma) = 4\sigma + 8(1-\sigma) \qquad (5-2)$$

where σ is the probability with which player 1 plays her pure strategy t. Equation $(5-2)$ is equivalent to

$$\sigma = \frac{2}{3}$$

So, in addition to the Nash equilibria in pure strategies, there is a Nash equilibrium in mixed strategies, in which player 1 plays (t, b) with probabilities $(\frac{2}{3}, \frac{1}{3})$,and in which player 2 plays his pure strategies (L, R) with probabilities $(\frac{2}{3}, \frac{1}{3})$.