GS/ECON 5010 APPLIED MICROECONOMICS

Answers to Midterm Exam October 2006

*Q*1. Show that a person's preferences can be represented by a continuous utility function, if the preferences are complete, transitive, continuous and strictly monotonic.

*A*1. This constructive proof is presented on pages 14 and 15 of *Jehle & Reny*. Start by assigning the value a to each consumption bundle $(a, a, \ldots, a)$, where a is any non–negative number. (So, for example, set $u(3, 3, 3, \ldots, 3) = 3$.)

Now take any other bundle $\mathbf{x}$. Let M be the biggest element in $\mathbf{x}$, and m be the smallest. For example, if $\mathbf{x} = (7, 3, 4, 9, 2)$, then $M = 9$ and $m = 2$. Strict monotonicity implies that $(M, M, M, \ldots, M) \succeq \mathbf{x}$, and that $\mathbf{x} \succeq (m, m, \ldots, m)$. Then continuity of preferences implies that there is some b, with $M \geq b \geq m$, such that

$$\mathbf{x} \sim (b, b, \ldots b)$$

[Formally : let b be the smallest number a such that $(a, a, \ldots, a) \succeq \mathbf{x}$; strict monotonicty and transitivity then imply that $(c, c, \ldots, c) \succ \mathbf{x}$ for any $c > b$, and that $\mathbf{x} \succ (d, d, \ldots, d)$ for any $d < b$; continuity of preferences then implies that $(b, b, \ldots, b) \sim \mathbf{x}$.]

Then let $u(\mathbf{x}) = b$. A utility function has thus been constructed, which is continuous, which is defined for each consumption bundle $\mathbf{x}$, and which represents the preferences, in that $u(\mathbf{x}) \geq u(\mathbf{z})$ if and only if $\mathbf{x} \succeq \mathbf{z}$ and $u(\mathbf{x}) > u(\mathbf{z})$ if and only if $\mathbf{x} \succ \mathbf{z}$.

*Q*2. Derive the Marshallian demand functions for goods 1 and 2, for a person whose preferences can be represented by the utility function

$$u(x_1, x_2) = 300 + \sqrt{x_1} + \sqrt{x_2}$$

With this utility function,

$$u_1(\mathbf{x}) = \frac{1}{2\sqrt{x_1}}$$

$$u_2(\mathbf{x}) = \frac{1}{2\sqrt{x_2}}$$

so that

$$MRS = \frac{u_1(\mathbf{x})}{u_2(\mathbf{x})} = \sqrt{\frac{x_2}{x_1}}$$

Utility maximization implies choosing a consumption bundle such that $MRS = p_1/p_2$, so that the consumer picks a bundle $\mathbf{x}$ such that

$$\sqrt{\frac{x_2}{x_1}} = \frac{p_1}{p_2}$$

or

$$x_2 = [\frac{p_1}{p_2}]^2 x_1 \qquad (2-1)$$

Plugging $(2-1)$ into the consumer's budget constraint $p_1x_1 + p_2x_2 = y$ implies that

$$p_1x_1 + p_2[\frac{p_1}{p_2}]^2 x_1 = y$$

or

$$p_1(1 + \frac{p_1}{p_2})x_1 = y \qquad (2-2)$$

Implying that

$$x_1 = \frac{p_2}{p_1}\frac{1}{p_1 + p_2}y \qquad (2-3)$$

Equation $(2-3)$ is the Marshallian demand function for good 1. Substituting back into $(2-1)$ then implies that

$$x_2 = \frac{p_1}{p_2}\frac{1}{p_1 + p_2}y \qquad (2-4)$$

is the Marshallian demand function for good 2.

(Or, since here the preferences are CES, with $\rho = 0.5$, you can plug in equations $(E10)$ and $(E11)$ from page 26 of *Jehle & Reny* to get an equivalent representation.)

*Q*3. Suppose that a person's utility of wealth function was $U(W) = aW - bW^2$ where $a > 0$, $b > 0$ (for wealth $W < a/2b$).

(*a*) What is the person's coefficient of absolute risk aversion, and her coefficient of relative risk aversion?

(*b*) If this person had a fixed amount of wealth to allocate between an asset with a certain rate of return r_0, and another "risky" asset with a stochastic return, how would the amount she invested in the risky asset vary with her initial wealth?

*A*3. Straightforward differentiation shows that

$$U'(W) = a - 2bW$$

$$U''(W) = -2b$$

so that

$$R_a(W) \equiv -\frac{U''(W)}{U'(W)} = \frac{2b}{a - 2bW}$$

$$R_r(W) \equiv -\frac{U''(W)W}{U'(W)} = \frac{2bW}{a - 2bW}$$

It then follows that

$$R'_a(W) = \frac{4b^2}{(a - 2bW)^2} > 0$$

so that the person's preferences exhibit increasing absolute risk aversion. The person's preferences must also exhibit increasing relative risk aversion, since $R'_r(W) > 0$ whenever $R'_a(W) > 0$.

When a person gets to allocate a fixed amount of wealth between a safe asset and a risky asset, the amount she chooses to invest in the risky asset will be an increasing function of her initial wealth if and only if her utility–of–wealth function exhibits decreasing absolutae risk aversion (DARA). Since $R'_a(W) > 0$ here, the amount she chooses to invest in the risky asset will be a **decreasing** function of her initial wealth.

But the wealth allocation problem can be solved explicitly here. The person wants to pick an investment X in the risky asset so as to maximize her expected utility

$$a[(1+r_0)W_0 + (Er - r_0)X] - bE\{[(1+r_0)W_0 + (r - r_0)X]^2\}$$

where W_0 is her initial wealth, r is the (stochastic) return to the risky asset, and Er is its expected value.

Her first–order condition for the maximization of her expected utility is

$$a[Er - r_0] - 2bE\{[(1+r_0)W_0 + (r - r_0)X](r - r_0)\} = 0$$

or

$$X = \frac{[a - 2b(1+r_0)W_0](Er - r_0)}{2bE\{(r - r_0)^2\}}$$

so that her total investment X in the risky asset declines with her initial wealth W_0 :

$$\frac{\partial X}{\partial W_0} = \frac{(1+r_0)(Er - r_0)}{E\{(r - r_0)^2\}} < 0$$

*Q*4. (*a*) What is a homothetic production function?

(*b*) What form will the firm's cost function take, if its production function is homothetic?

(*c*) What form will the firm's conditional input demand functions take, if its production function is homothetic?

*A*4. A homothetic production function is a production function $y = f(\mathbf{x})$ which can be written in the form

$$f(\mathbf{x}) = \Phi(g(\mathbf{x}))$$

where $\Phi(\cdot)$ is some increasing function mapping real numbers into real numbers, and where $g(\mathbf{x})$ is a function which is homogeneous of degree 1, that is a function for which

$$g(kx_1, kx_2, \ldots, kx_n) = kg(x_1, x_2, \cdots, x_n)$$

for any $k > 0$, and any vector $\mathbf{x}$ of inputs.

Since $f(\cdot)$ is a monotonically increasing transformation of $g(\cdot)$, both production functions have the same shape of isoquants : only the levels of output associated with the isoquants differ.

So suppose that $\mathbf{x}$ minimizes the cost of producing y for the production function $g(\cdot)$. That means that the isoquant corresponding to the output level y for the production function $g(\cdot)$ is the same as the isoquant corresponding to the output level $\Phi(y)$ for the production function $f(\cdot)$. If $\mathbf{x}$ minimizes the cost of the output level y for the production function $g(\cdot)$, then $\mathbf{x}$ also minimizes the cost of the output level $\Phi(y)$ for the production function $g(\cdot)$.

That means that $C(\mathbf{w}, \Phi(y)) = D(\mathbf{w}, \mathbf{y})$ for any input price vector $\mathbf{w}$, and any output level y, where $C(\mathbf{w}, y)$ is the cost function associated with the homothetic production function $f(\cdot)$ and $D(\mathbf{w}, y)$ is the cost function associated with the constant–returns–to–scale production function $g(\cdot)$.

Constant returns to scale imply that if $\mathbf{x}$ minimizes the cost of producing y units of output (with the production function $g(\cdot)$) then $k\mathbf{x}$ minimizes the cost of producing an output level of ky. So $D(\mathbf{w}, ky) = kD(\mathbf{w}, y)$.

Supppose that $\Phi(z) = 1$. Then

$$C(\mathbf{w}, 1) = D(\mathbf{w}, z)$$

Take any other output level y. Then find $z' = kz$ such that $\Phi(z') = y$. Then

$$C(\mathbf{w}, y) = D(\mathbf{w}, z') = kD(\mathbf{w}, z) = kC(\mathbf{w}, 1)$$

That means that the cost function$C(\mathbf{w}, y)$ associated with the homothetic production function $f(\cdot)$ can be written

$$C(\mathbf{w}, y) = h(y)C(\mathbf{w}, 1)$$

where $h(y) = k = \Phi^{-1}(y)/\Phi^{-1}(1)$ is an increasing function of y.

Shepherd's Lemma then implies that the conditional demand for any input i (for the homothetic production technology $f(\cdot)$) obeys

$$x_i(\mathbf{w}, y) = \frac{\partial C(\mathbf{w}, y)}{\partial w_i} = h(y)\frac{\partial C(\mathbf{w}, 1)}{\partial w_i} = h(y)x_i(\mathbf{w}, 1)$$

[The correct answers to parts (b) and (c) did not need all that derivation or proof above : simply stating that $C(\mathbf{w}, y) = h(y)C(\mathbf{w}, 1)$ for some increasing function $h(y)$, and that $x_i(\mathbf{w}, y) = h(y)x_i(\mathbf{w}, 1)$ would be good enough.]