GS/ECON 5010 section “B” answers to midterm exam

$Q1$. Are the preferences represented by the utility function

$$U(x_1, x_2, x_3) = x_1 - \frac{1}{x_2} + (x_3)^2$$

strictly monotonic? Convex? Explain.

$A1$. The preferences are strictly monotonic because the partial derivatives are all positive for any consumption vector $\mathbf{x} >> 0$:

$$\frac{\partial U}{\partial x_1} = 1 \qquad (1-1)$$

$$\frac{\partial U}{\partial x_2} = \frac{1}{(x_2)^2} \qquad (1-2)$$

$$\frac{\partial U}{\partial x_3} = 2x_3 \qquad (1-3)$$

But the term $(x_3)^2$ suggests that preferences may not be convex. To show that preferences indeed are not convex, a single counter–example is a all that is needed. Here

$$U(\mathbf{x}^1) = U(4, 1, 1) = 4 \qquad (1-4)$$

$$U(\mathbf{x}^2) = U(1, 1, 2) = 4 \qquad (1-5)$$

But the consumption bundle which is halfway between $(4, 1, 1)$ and $(1, 1, 2)$ is $(2.5, 1, 1.5)$ and

$$U(2.5, 1, 1.5) = 3.75 < 4 \qquad (1-6)$$

so that $U(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) < U(\mathbf{x}^1) = U(\mathbf{x}^2)$ when $t = 0.5$, demonstrating that preferences are not convex.

$Q2$. Define the term “certainty equivalent” to a gamble, and show that the certainty equivalent to a gamble must be less than the expected value of the gamble for a risk–averse (von Neumann Morgenstern) expected utility maximizer.

$A2$. The certainty equivalent CE to a gamble is the amount of wealth such that

$$U(CE) = p_1 U(W_1) + p_2 U(W_2) + \cdots + p_n U(W_n) \tag{2-1}$$

if the gamble g is defined as

$$g \equiv (p_1 \circ W_1, p_2 \circ W_2, \ldots, p_n \circ W_n)$$

If the person is risk averse, then her utility–of–wealth function $U(\cdot)$ must be concave. And by definition, for **any** concave function,

$$U(p_1 W_1 + p_2 W_2 + \cdots + p_n W_n) > p_1 U(W_1) + p_2 U(W_2) + \cdots + p_n U(W_n) \tag{2-2}$$

So equations $(2-1)$ and $(2-2)$ together imply that

$$U(p_1 W_1 + p_2 W_2 + \cdots + p_n W_n) > U(CE) \tag{2-3}$$

and the fact that $U' > 0$ and $(2-3)$ then imply that

$$p_1 W_1 + p_2 W_2 + \cdots + p_n W_n > CE \tag{2-4}$$

which means the expected value of the gamble must exceed the certainty equivalent.

$Q3$. What is the cost function for a firm with a production function

$$f(x_1, x_2) = x_1 + \log(1 + x_2)$$

(where "log" denotes the natural logarithm)?

$A3$. Cost minimization requires that

$$f_1/f_2 = w_1/w_2 \tag{3-1}$$

where f_i refers to the partial derivative of the production function with respect to the quantity of input i. Here, equation $(3-1)$ becomes

$$1 + x_2 = \frac{w_1}{w_2} \tag{3-2}$$

Substituting for x_2 from $(3-2)$ into the production function,

$$y = f(x_1, x_2) = x_1 + \log w_1 - \log w_2 \qquad (3-3)$$

so that the conditional input demand for input 1 can be written

$$x_1(w_1, w_2, y) = y - \log w_1 + \log w_2 \qquad (3-4)$$

Meanwhile, $(3-2)$ can be written

$$x_2(w_1, w_2, y) = \frac{w_1}{w_2} - 1 \qquad (3-5)$$

The cost function $C(w_1, w_2, y)$ must equal $w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y)$, or

$$C(w_1, w_2, y) = w_1 y - w_1 \log w_1 + w_1 \log w_2 + w_1 - w_2 \qquad (3-6)$$

[Equation $(3-5)$ is valid only if $w_1 \geq w_2$; if $w_2 > w_1$ then the firm should use only input #1, so that $x_1(w_1, w_2, y) = y$ and $C(w_1, w_2, y) = w_1 y$.]