Answers to a few Questions on Previous Final Exams

F2005, Q2 : If a person's utility–of–wealth function were

$$u(W) = A - \frac{B}{W}$$

how would her coefficients of relative risk aversion **and** absolute risk aversion vary with her wealth?

If she had to divide her wealth between an asset with a certain return, and an asset with a risky return, how would her asset allocation depend on the total amount of wealth that she had to invest?

answer : Since

$$u'(W) = \frac{B}{W^2} \tag{1}$$

so that

$$u''(W) = -\frac{2B}{W^3}$$

therefore, the coefficient of relative risk aversion,

$$R_R(W) \equiv -\frac{u''(W)W}{u'(W)} = 2$$

so that the person has a **constant** coeficient of relative risk aversion.

With a constant coefficient of relative risk aversion, the **proportion** of the person's wealth which is allocated among different assets will not vary with her wealth.

To see this, suppose that the person has a fixed initial wealth W_0, which she can allocate among n different assets. The net return to asset i is denoted r_i. This return is random [unless the asset has a sure return, which is a special case of a random return].

Let x_i be the **proportion** of the person's wealth which she allocates to asset i, so that

$$\sum_{i=1}^{n} x_i = 1 \tag{2}$$

The person's end–of–period wealth is

$$W_1 = [\sum_{i=1}^{n}(1 + r_i)x_i]W_0$$

She wishes to maximize her expected end–of–period utility

$$Eu(W_1) = Eu([\sum_{i=1}^{n}(1 + r_i)x_i]W_0)$$

subject to the constraint (2). Given the form of the utility function,

$$Eu(W_1) = E(A - \frac{B}{[\sum_{i=1}^{n}(1 + r_i)x_i]W_0})$$

Since A, B and W_0 are non–random constants,

$$Eu(W_1) = A - \frac{B}{W_0} E(\frac{1}{\sum_{i=1}^{n}(1+r_i)x_i})$$

so that maximization of her expected utility is equivalent to minimization of

$$E(\frac{1}{\sum_{i=1}^{n}(1+r_i)x_i}) \tag{4}$$

subject to the constraint (2).

But the person's initial wealth W_0 does not appear in expression (4), nor in the constraint (2). That means that the person's optimal $(x_1, x_2, \ldots, x_n)$, the proportions of her wealth to put in each of the assets, which minimize expression (4) subject to constraint (2), do not depend on her initial wealth W_0.

F2004, Q2 : If a person had a fixed amount of wealth W_0 to allocate between a safe asset and a risky asset, how would her expected–utility–maximizing portfolio vary with her wealth W_0, and with the parameter α, if her utility of wealth could be written

$$U(W) = \frac{1}{1-\alpha} W^{1-\alpha}$$

where α is a positive parameter?

answer : Here

$$U'(W) = W^{-\alpha}$$

$$U''(W) = -\alpha W^{-(1+\alpha)}$$

So that

$$R_R(W) = \alpha$$

and the coefficient of relative risk aversion is constant (just as in the answer to 2005:Q2 above).

The answer to 2005:Q2 also shows that the person's investment in the safe and risky assets would be proportional to her wealth.

To find out what those proportions are, and how they vary with α, let x denote the proportion of her wealth placed in the risky asset, so that her end–of–period wealth is

$$W_1 = (1+r)xW_0 + (1+r_0)(1-x)W_0 = (1+r_0)W_0 + (r-r_0)xW_0$$

where r is the random return to the risky asset, and r_0 is the certain return to the safe asset.

The person's expected end–of–period utility is then

$$E\{U(W_1)\} = \frac{1}{1-\alpha} W_0^{1-\alpha} E\{((1+r_0) + (r-r_0)x)^{1-\alpha}\} \tag{1}$$

Choosing x to maximize expression (1) with respect to x implies a first–order condition for optimality of

$$E\{(r-r_0)[(1+r_0)+(r-r_0)x]^{-\alpha}\}=0 \tag{2}$$

To find how x varies with α, differentiate equation (2) with respect to x and α : since (2) can be written

$$F(x,\alpha)=0$$

therefore

$$\frac{\partial x^*}{\partial \alpha}=-\frac{F_\alpha}{F_x}$$

where x^* is the person's optimal proportion of wealth to be placed in the risky asset. From (2),

$$F_x=-\alpha E\{(r-r_0)^2[(1+r_0)+(r-r_0)x]^{-(1+\alpha)}\}<0$$

Since

$$b^a \equiv ae^{\log b}$$

for any a,b, therefore

$$F_\alpha=\frac{\partial}{\partial \alpha}E\{(r-r_0)[(1+r_0)+(r-r_0)x]^{-\alpha}\}=-E\{(r-r_0)[(1+r_0)+(r-r_0)x]\}<0$$

so that

$$\frac{\partial x^*}{\partial \alpha}=-\frac{F_\alpha}{F_x}<0$$

Not surprisingly, a higher coefficient of relative risk aversion leads to the person wanting to put a smaller fraction of her wealth in the risky asset.

W2005, Q2 : If a person with a fixed amount of wealth W_0 could allocate that wealth between a safe asset, offering a certain rate of return $r \geq 0$, and a risky asset, offering the return $r_g > r$ with some probability π, and the return $r_b < r$ with probability $1-\pi$, how much wealth should she invest in the safe asset, and how much in the risky asset, if her utility–of–wealth function is $u(W)=A-e^{-\alpha W}$?

answer Here the person has a constant coefficient of **absolute** risk aversion. Since

$$u'(W)=\alpha e^{-\alpha W}$$

and

$$u''(W)=-\alpha^2 e^{-\alpha W}$$

therefore

$$R_A(W)=\alpha$$

With a constant coefficient of absolute risk aversion, the person will put the same **absolute** amount of wealth in the risky asset, regardless of her income.

What is that constant amount? If the person invests X dollars in the risky asset, and $W_0 - X$ in the safe asset, then her end–of–period wealth W_1 will be $(1+r)W_0 + (r_g - r)X$ with probability π, and $(1+r)W_0 - (r - r_b)X$ with probability $1 - \pi$. Therefore, her expected utility of end–of–period wealth is

$$Eu = A - \pi \exp\left(-\alpha[(1+r)W_0 + (r_g - r)X]\right) - (1-\pi)\exp\left(-\alpha[(1+r)W_0 - (r - r_b)X]\right) \tag{1}$$

where "exp" denotes the exponential function : $\exp a \equiv e^a$. Since $e^{a+b} = e^a e^b$, expression (1) can be written

$$Eu = A - \exp\left(-\alpha(1+r)W_0\right)[\pi \exp\left(-\alpha(r_g - r)X\right) + (1-\pi)\exp\left(\alpha(r - r_b)X\right)] \tag{2}$$

Maximizing (2) with respect to X is the same as maximizing

$$\pi \exp\left(-\alpha(r_g - r)X\right) + (1-\pi)\exp\left(\alpha(r - r_b)X\right) \tag{3}$$

with respect to X.

Differentiating (3) with respect to X, and setting it equal to 0,

$$(r_g - r)\pi \exp\left(-\alpha(r_g - r)X\right) = (r - r_b)\exp\left(\alpha(r - r_b)X\right) \tag{4}$$

Taking natural logarithms of both sides of (4),

$$\log[\pi(r_g - r)] - \alpha(r_g - r)X = \log[(1-\pi)(r - r_b)] + \alpha(r - r_b)X$$

or

$$X = \frac{1}{\alpha(r_g - r_b)} \log\left[\frac{\pi(r_g - r)}{(1-\pi)(r - r_b)}\right] \tag{5}$$

which shows that total investment in the risky asset is independent of the person's initial wealth.

F2004, Q4 : What would be the equilibrium price and quantity in the long run, in a competitive industry in which there were many identical firms, each with the same long run total cost function

$$TC(q) = q^3 - 12q^2 + 60q$$

where q was the output of the firm, if the market demand curve for the output of the firms had the equation

$$Q = 540 - 5p$$

where Q was the total quantity demanded, and p the price of the good?

answer : Since

$$TC(q) = q^3 - 12q^2 + 60q$$

then

$$AC(q) = TC(q) = q^2 - 12q + 60 \tag{1}$$

Taking the derivative of (1),

$$AC'(q) = 2q - 12 \tag{2}$$

so that each firm's long–run average cost curve is U–shaped, decreasing in output for $0 < q < 6$, and increasing in output for $q > 6$.

In an industry with many identical firms, each with a U–shaped long–run average cost curve, in the long–run, each firm's output must equal the level of output q^* for which average cost is at a minimum. Only there is $AC = MC$: profit maximization by firms implies that $p = MC$, and zero profit for the marginal firm in the industry implies that $p = AC$.

Equation (2) implies that $q^* = 6$. This can be confirmed by calculating the marginal cost

$$MC(q) = TC'(q) = 3q^2 - 24q + 60 \tag{3}$$

Equations (2) and (3) imply that $MC(q) = AC(q)$ if and only if $2q^2 = 12q$, or $q = 6$.

If each active firm produces a level of output $q^* = 6$, then each active firm's marginal (and average) cost is

$$MC(q^*) = 3(6)^2 - 24(6) + 60 = 24$$

So the equilibrium price int he industry must be each firm's marginal (and average) cost, 24. If the price is 24, then aggregate demand is

$$Q = 540 - 5(24) = 420$$

In long–run equilibrium, 70 firms will be active, each producing an output of 6, at an average cost of 24, which equals the market price..

F2005, Q5 : What is the relation between the price elasticity of demand for the output of a single–price monopoly, and the amount which the mark–up of price above cost?

answer : If $P(Q)$ denotes the monopoly's inverse demand curve (that is, $P(Q)$ is the price it must charge, if it must charge the same price for each unit, if it wants to sell exactly Q units) then the firm's profits are

$$\pi(Q) = P(Q)Q - C(Q)$$

where $C(Q)$ is the monopoly's total cost function. Maximizing profit means choosing an output level such that

$$P'(Q)Q + P(Q) = C'(Q) \tag{1}$$

which can be written

$$\frac{P(Q) - C'(Q)}{P(Q)} = -\frac{P'(Q)Q}{P(Q)} \tag{2}$$

The right side of equation (2) is the inverse of the firm's own–price elasticity of demand :

$$\eta \equiv -D'(p)\frac{p}{D(p)}$$

where $D(p)$ is the equation of the firm's aggregate demand funtion. Since $P(Q)$ is the inverse function to $D(p)$ — $P(D(p)) \equiv p$, therefore $P'(Q) - \frac{1}{D'(P(Q))}$ so that

$$-\frac{P'(Q)Q}{P(Q)} = \frac{1}{\eta}$$

Therefore, equation (2) says that the firm's mark–up over marginal cost, expressed as a function of the final price, is $1/\eta$. Expressing this mark–up as a fraction of the cost, equation (2) implies that

$$\frac{p - MC}{MC} = \frac{1}{\eta - 1} \tag{3}$$

if the monopoly chooses its price so as to maximize profits.

F2004, Q5 : What would be the equilibrium output in a Cournot duopoly, if there were a price ceiling $\bar{p}$ imposed on the market?

answer Let $P(Q)$ be the inverse demand function for the industry, that is $P(Q)$ is the price that would induce consumers to buy (in aggregate) Q units of the good.

If firm 1 produced q_1 units of the good, and firm 2 produced q_2, then the price for which they could sell their output would be the **minimum** of $\bar{p}$ and $P(Q)$. Let $\bar{Q}$ be the total output level fat which the price ceiling begins to bind : that is, define $\bar{Q}$ by $P(\bar{Q}) = \bar{p}$.

The net profit that firm 1 earns is $pq_1 - C(q_1)$, where $C(\cdot)$ is its total cost function. Given the other firm's output q_2, this profit is

$$\bar{p}q_1 - C(q_1) \quad \text{if} \quad q_1 < \bar{Q} - q_2$$

$$P(q_1 + q_2)q_1 - C(q_1) \quad \text{if} \quad q_1 > \bar{Q} - q_2$$

So the marginal added profit from producing a little more output decreases discontinuously for firm 1, from

$$\bar{p} - MC$$

when $\quad q_1 < \bar{Q} - q_2$ to

$$p + P'(Q)q_1 - MC$$

when $\quad q_1 > \bar{Q} - q_2$

This discontinuous fall in the marginal profitability of an increase in output means that $q_1 = \bar{Q} - q_2$ will be firm 1's best response to firm 2, whenever

$$\bar{p} > MC(\bar{Q} - q_2) > \bar{p} + P'(\bar{Q})(\bar{Q} - q_2) \tag{1}$$

If condition (1) holds, then firm 1 would not want to decrease output below $\bar{Q} - q_2$, since the (government–mandated) price $\bar{p}$ exceeds the marginal cost, but it would not want to increase output, since the marginal revenue of further output increases (for which the price falls below the ceiling) is less than the marginal cost.

Suppose, for simplicity, that the marginal cost of production were a constant c, the same for each firm. Suppose as well that there is a unique Cournot equilibrium when there is no price ceiling, the output levels $q_1 = q_2 = q^c$ at which

$$P(2q^c) + P'(2q^c)q^c = c \tag{2}$$

If the price ceiling $\bar{p}$ exceeds $P(2q^c)$, then it is irrelevant : the two firms will each choose $q_1 = q_2 = q^c$: here neither firm would want to decrease output below q^c.

But if $P(2q^c) > \bar{p} > c$, then $q_1 = q_2 = q^c$ cannot be a Cournot equilibrium, since then the price ceiling binds, and each firm would want to increase output.

What would be the Cournot equilibrium? : **any** pair of output levels (q_1, q_2) such that i $q_1 + q_2 = \bar{Q}$ ii $\bar{p} + P'(\bar{Q})q_i < c \quad i = 1, 2$ Condition i ensures the price ceiling binds : since $\bar{p} > c$, neither firm will want to decrease output. Condition ii ensures that neither firm will want to increase output.

If $P(2q) + P'(2q)q$ is a decreasing function of q, then $P(\bar{Q}) + P'(\bar{Q})\frac{\bar{Q}}{2} > P(2q^c) + P'(2q^c)q^c = c$ whenever $P(2q^c) > \bar{p}$. So, for example, $q_1 = q_2 = \frac{\bar{Q}}{2}$ must satisfy conditions (i) and (ii) whenever $P(2q^c) > \bar{p}$. But so might other (q_1, aq_2) combinations with $q_1 + q_2 = \bar{Q}$, so long as neither q_1 nor q_2 were so small that condition ii no longer held.

In summary, a price ceiling which is below the price prevailing in the (unconstrained) Cournot equilibrium will be effective. If $\bar{p} < P(2q^c)$ then the price ceiling leads to an equilibirum in which aggregate output exceeds $2q^c$, and in which the legal price ceiling binds : the lower the price ceiling, the higher the output (as long as the price ceiling exceeds the marginal cost.

F2005, Q10. Suppose a prize is to be auctioned off, using a first–price, sealed bid auction. There are 2 buyers. Each buyer's value for the prize is an independent draw from the same distribution, $U[0, 1]$ (that is, the uniform distribution over the interval $[0, 1]$).

What is the expected revenue from the auction?

answer : Two ways to get the same answer :

i Using the Revenue equivalence Theorem, the first– and second–price auctions have the same expected revenue, if buyers bids are independent draws.

In a seond–price (or English) auction, bidders have dominant strategies, to bid their true values.

Thereore, each bidder's bid will be an independent draw from the $U[0, 1]$ distribution, in a scond–price auction.

The price actually received in a second price auction is the **lower** of the two buyers' valuations.

What is the probability that both buyers' valuations are above v, for some $0 < v \leq 1$? That probability is $[1 - F(v)]^2$, where $F(v)$ is the distribution function for each buyer's valuation.

So, in general, if there are two bidders in an English auction, each with valuation drawn from the distribution $F(\cdot)$, then the probability that the lowest bid is greater than v is $[1-F(v)]^2$, which means that the probability that the lowest bid is less than v is

$$G(v) = 1 - [1 - F(v)]^2$$

So $G(v)$ is the distribution function for the winning bid in the two–person English auction. Its density function is

$$g(v) \equiv G'(v) = 2[1 - F(v)]f(v)$$

which means that the expected revenue for the auctioneer is

$$ER = \int_0^1 vg(v)dv = 2\int_0^1 v[1 - F(v)]f(v)dv$$

Here the distribution is uniform, so that $F(v) = v$, and $f(v) = 1$. That menas that

$$EV = 2\int_0^1 (v - v^2)dv$$

Since $v - v^2$ is the integral of $v^2/2 - v^3/3$, therefore

$$ER = 2[v^2/2 - v^3/3]|_0^1 = 2(1/2 - 1/3) = \frac{1}{3}$$

ii In a first–price auction with 2 people (whose values are independent draws from the same distribution), each player's optimal bid $b(v)$ is

$$b(v) = \frac{1}{F(v)}\int_0^v xf(x)dx$$

if her value is v, where $F(\cdot)$ is the distribution function for bidders' values, and $f(\cdot)$ is the density function. Here $F(v) = v$, nd $f(v) = 1$, so that

$$b(v) = \frac{1}{v}\int_0^v vdx = \frac{v}{2}$$

That is, with uniform values, each bidder should bid half her value if there are 2 bidders.

What is the probability that the winning bidder has a value of v of less? This event will occur with probability $[F(v)]^2$, since the winning bidder will have value of v or less only if **both** bidders have values of v or less. Therefore, the distribution function $H(\cdot)$ for the value of the high bidder is

$$H(v) = [F(v)]^2 = v^2$$

with associated density function

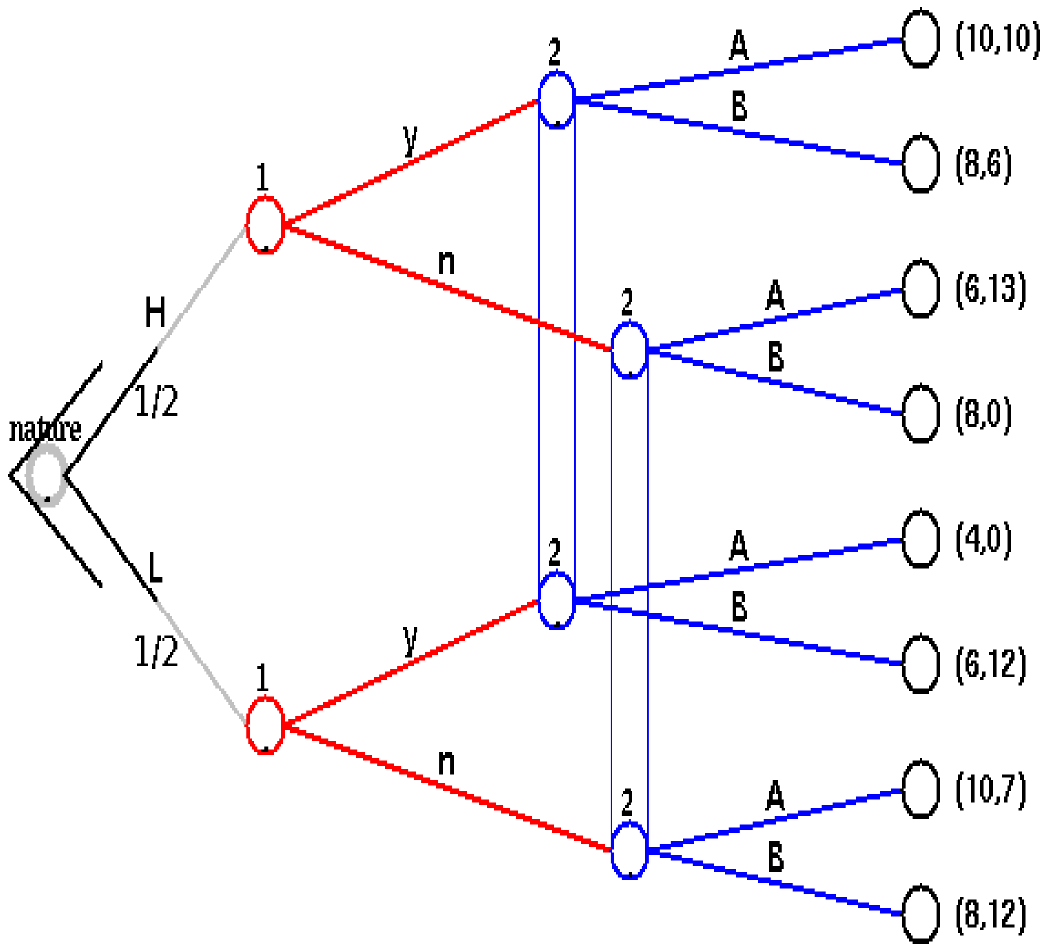
$$h(v) = 2f(v)F(v) = 2v$$

If the winning bidder has value v, she will bid $b/2$. So the expected revenue to the auctioneer is

$$ER = \int_0^1 \frac{v}{2} h(v)dv = \int_0^1 v^2 dv = \frac{1}{3}v^3|_0^1 = \frac{1}{3}$$

F2005, Q9. The accompanying figure shows a two–person game of incomplete information, in which "nature" moves first, choosing "H" or "L", each with probability 1/2, in which player 1 moves next, choosing "y" or "n" after she has observed nature's move, and in which player 2 moves last, choosing "A" or "B" after he has observed player 1's move, but without having observed nature's move.

Find every perfect Bayesian equilibrium to this game.



answer First, note that player 1 has a strictly dominant strategy if she is of type "L" : the worst that can haappen if she plays "n" is 8, whereas the best that can happen if she plays "y" is 6. So, if she is type "L" she will always play "n".

Second, she has a weakly dominant strategy if she is of type "H" : then "y" is strictly better than "n" for her if player 2 plays "A", and just as good as "n" if player 2 plays "B".

So a candidate for equilibrium is having player 1 play her weakly and strictly dominant strategies : "y" if she is of type "H" and "n" if she is of type "L". In this case, player 1's actions reveal her type perfectly : if player 2 sees the action "y" he knows that player 1 is of type "H" and if he sees "n" he knows player 1 is of type "L".

In this case, his best reaction is to play "A" if he sees "y" (since then he knows he is at the top decision node in the diagram), and "B" if he sees "n" (since then he knows he is at the bottom decision node in the diagram).

So the actions and beliefs described above constitute a perfect Bayesian equilibrium equilibrium : 1 plays "y" for sure if she is type "H", and "n" for sure if she is type "L" ; 2 believs that player 1 is of type "H" for sure if she plays "y" and of type "L" for sure if she plays "n" ; 2 plays "A" if 1 plays "y", and "B" if 1 plays "n".

Are there any other equilibria? In any perfect Bayesian equilibrium, player 1 must play "n" if she is of type "L", since that is a dominant strategy. But she might play "n" if she were of type "H" — but only if she were absolutely sure that player 2 would play "B" for sure (whatever player 1 did). In this case, the type–"H" player 1 would get 8 as a payoff, whether she played "y" or "n".

Would player 2 be willing to play "B" if she saw player 1 play "n"? His his expected payoff to "A" is $13\alpha + 7(1-\alpha)$ if he plays "A", and $12(1-\alpha)$ if he plays "B", where α is his belief that player 1 is of type "H", given that she chose "n". If $\alpha > 5/18$, she will prefer playing "A" to playing "B", in response to "n".

So there cannot be a perfect Bayesian equilibrium in which player 1 played "n" all of the time, whether she is of type "H" or type "L" : if she played "n" all the time, player 2 would believe that the probability α that she is of type "H" (given that she has played "n") would be the a priori probability 0.5 that nature has chosen her to be "'H". If player 2's belief $alpha = 0.5$, then he will prefer to play "A" rather than "B". And if he plays "A" in response to "n", player 1 will not want to play "n" if she is of type "H".

Finally, could there be a perfect Bayesian equilibrium in which player 1 mixes between "y" and "n" if she is of type "H"? In that case, player 2 must believe that she is of type "H" for sure, if she happens to play "y" : player 1 would never play "y" if she were of type "L", so that Bayes's rule says $P(H|y) = 1$ if $P(y|H) > 0$ and $P(y|L) = 0$.

So player 2 would know that player 1 is of type "H" if he saw "y". That means that he would always play "A" in response to "y".

But then player 1 would be unwilling to mix between "y" and "n" if she were of type "H" : "y" gives her 10 for sure, and 'n" gives her no more than 8.

So in any perfect Bayesian equilibrium, player 1 must play "y" for certain if she is of type

"H", and "n" for certain if she is of type "L". That means that the perfect Bayesian equilibrium described above is the unique perfect Bayesian equilibrium to the game.