Homogeneity of Degree α

a function $f : \Re^n \to \Re$ is homogeneous of degree α if

$$f(t\mathbf{x}) = t^{\alpha} f(\mathbf{x}) \tag{1}$$

for any input vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and any positive scalar t

CES :

$$f(\mathbf{x}) \equiv (a_1 x_1^{\rho} + a_2 x_2^{\rho} + \cdots + a_n x_n^{\rho})^{\mu/\rho}$$

is homogeneous of degree μ

Theorem 3.1 (generalized) : if $f : \Re^n \to \Re$ is continuous, strictly monotonic, and quasi–concave, and f is also homogeneous of degree $\alpha \leq 1$, then f is not just quasi–concave, it's **concave**

Returns to Scale

if a production function is homogeneous of degree α, then it exhibits

increasing returns to scale if $\alpha > 1$

constant returns to scale if $\alpha = 1$

decreasing returns to scale if $\alpha < 1$

but..not every production function is homogeneous of degree α

so a "local" measure of returns to scale, $\mu(\mathbf{x})$ is defined in the following manner :

elasticity of output with respect to input i :

$$\mu_i(\mathbf{x}) \equiv \frac{\partial f(\mathbf{x})}{\partial x_i}\frac{x_i}{f(\mathbf{x})} \qquad (2)$$

$$\mu(\mathbf{x}) \equiv \sum_i \mu_i(\mathbf{x}) \tag{3}$$

with CES technology,

$$\mu_i(\mathbf{x}) = \mu a_i(\sum_j a_j x_j^\rho)^{\mu/(\rho-1)} x_i^{\rho-1} \frac{x_i}{[\sum_j a_j x_j^\rho]^{\mu/\rho}}$$

which equals

$$\mu \frac{a_i x_i^\rho}{\sum_j a_j x_j^\rho}$$

implying that

$$\mu(\mathbf{x}) = \mu$$

Some Properties of CRS Production Functions

$f(\mathbf{x})$ is homogeneous of degree 1, then it can be written in the form

$$f(\mathbf{x}) = x_1 g(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1})$$

where the function g is a function of the $n-1$ input ratios $\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}$. In fact,

$$g(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1}) = f(1, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_n}{x_1})$$

Since $\mu(\mathbf{x}) = \sum_j \mu_j(\mathbf{x})$, then

$$\mu(\mathbf{x}) = \frac{\sum_j f_j(\mathbf{x}) x_j}{f(\mathbf{x})}$$

so that constant returns to scale, which means that μ equals 1, implies that

$$\sum_j f_j(\mathbf{x}) x_j = f(\mathbf{x})$$