Profit Maximization : Cobb–Douglas Example

$$f(\mathbf{x}) = \Pi_{i=1}^{n} x_i^{a_i} \tag{1}$$

where

$$a \equiv a_1 + a_2 + \cdots + a_n < 1$$

$$\frac{\partial f}{\partial x_i} = \frac{a_i}{x_i} f(\mathbf{x}) \tag{2}$$

first–order condition for cost minimization

$$\frac{f_i}{f_j} = \frac{w_i}{w_j}$$

from equation (2),

$$\frac{w_i}{w_1} = \frac{a_i}{a_1}\frac{x_1}{x_i} \tag{3}$$

or

$$x_i = \frac{a_i}{w_i}\frac{w_1}{a_1}x_1 \quad i = 1, 2, \cdots, n \tag{4}$$

to produce output of y,

$$\Pi_{i=1}^{n} x_i^{a_i} = y \tag{5}$$

substituting from (4),

$$y = (\frac{w_1}{a_1})^{a_1} \left[\Pi_{i=1}^{n} a_i^{a_i}\right] \left[\Pi_{i=1}^{n} w_i^{-a_i}\right] x_1^a \tag{6}$$

or

$$y = (\frac{w_1}{a_1})^{a_1} \frac{A}{W} x_1^a \tag{7}$$

where

$$A \equiv \Pi_{i=1}^{n} a_i^{a_i} \quad ; \quad W \equiv \Pi_{i=1}^{n} w_i^{a_i}$$

conditional input demands

re–arranging (6) yields

$$x_1^c(\mathbf{w}, y) = \frac{a_1}{w_1}(\frac{W}{A})^{1/a} y^{1/a} \tag{8}$$

and substituting back into (4)

$$x_i^c(\mathbf{w}, y) = \frac{a_i}{w_i}(\frac{W}{A})^{1/a} y^{1/a} \tag{9}$$

cost function : cost of conditional input demands

$$C(\mathbf{w}, y) = a(\frac{W}{A})^{1/a} y^{1/a} \tag{10}$$

profit maximization

maximize

$$py - C(\mathbf{w}, y) \tag{11}$$

where $C(\mathbf{w}, y)$ is defined by equation (10)

first–order condition

$$p - (\frac{W}{A})^{1/a} y^{1/a-1} = 0 \tag{12}$$

second–order condition

$$\frac{1-a}{a}(\frac{W}{A})^{1/a} y^{1/a-2} > 0 \tag{13}$$

holds if $a < 1$

equation (12) can be re–written

$$y = p^{a/(1-a)}\Big(\frac{W}{A}\Big)^{-1/(1-a)} \tag{14}$$

which is the supply curve for the firm

profit function

$\pi(p, \mathbf{w})$ is the maximized value of $py - C(\mathbf{w}, y)$, or

$$\pi(p, \mathbf{w}) = py(p, \mathbf{w}) - C(\mathbf{w}, y(p, \mathbf{w})) \tag{15}$$

Substituting from (14), $\pi(p, \mathbf{w})$ equals

$$pp^{a/(1-a)}\Big(\frac{W}{A}\Big)^{-1/(1-a)} - a\Big(\frac{W}{A}\Big)^{1/a}\Big[p^{a/(1-a)}\Big(\frac{W}{A}\Big)^{-1/(1-a)}\Big]^{1/a} \tag{16}$$

which simplifies to

$$\pi(p, \mathbf{w}) = (1-a)p^{1/(1-a)}\Big(\frac{W}{A}\Big)^{-1/(1-a)} \tag{17}$$

unconditional input demands

one way : substitute from supply function (14) into conditional factor demands (9)

$$x_i^u(p, \mathbf{w}) = x_i^c(\mathbf{w}, y(p, \mathbf{w}))$$

here that implies

$$x_i^u(p, \mathbf{w}) = \frac{a_i}{w_i}\left(\frac{W}{A}\right)^{1/a}[y(p, \mathbf{w})]^{1/a} \qquad (18)$$

or (substituting from (14)),

$$x_i^u(p, \mathbf{w}) = \frac{a_i}{w_i}\left(\frac{W}{A}\right)^{1/a}\left[p^{a/(1-a)}\left(\frac{W}{A}\right)^{-1/(1-a)}\right]^{1/a} \qquad (19)$$

which can be simplified to

$$x_i^u(p, \mathbf{w}) = \frac{a_i}{w_i}p^{1/(1-a)}\left(\frac{W}{A}\right)^{-1/(1-a)} \qquad (20)$$

alternate method : use Hotelling's Lemma (2) ; the unconditional factor demands are the negatives of the partial derivatives of the profit function (17) with respect to the input prices w_i