Nash Equilibrium

terminology : let S be player 1's set of strategies (e.g. $S = \{t, m, b\}$),

and let T be player 2's set of strategies

let $u^1(s,t)$ be player 1's payoff, when she plays $s \in S$, and when player 2 plays a strategy $t \in T$

strategy s^* is called a **best response** for player 1, to player 2's strategy t if

$$u^1(s^*, t) \geq u^1(s, t) \quad \text{for} \quad \text{all} \quad s \in S$$

in other words : if player 2 plays t, then s^* will be a best response for player 1 if s^* maximizes player 1's payoff along the column corresponding to strategy t for player 2

or (just) : s^* is a best response for 1 to t if s^* is the best strategy for 1 to choose, if 2 chooses t

definition : a **pair** of strategies (s^N, t^N) is a pure strategy **Nash equilibrium** for the 2–player game if (and only if)

s^N is 1's best response to t^N

t^N is 2's best response to s^N

so example 9 is a game which has no weakly dominated players for either player, but which does have a Nash equilibrium

$$\begin{pmatrix} 1\backslash 2 & L & C & R \\ & & & \\ t & (1,6) & (2,3) & (4,5) \\ b & (0,2) & (1,4) & (5,3) \end{pmatrix}$$

(t, L) is the (only) Nash equilibrium to the above game

game 2 (the coordination game) also has no dominated strategies, but it does have a Nash equilibrium in pure strategies : 2 of them actually ; so does this example, example 11

$$\begin{pmatrix} 1\backslash 2 & L & R \\ & & \\ t & (6,1) & (0,0) \\ b & (5,9) & (1,10) \end{pmatrix}$$

Nash equilibria (in pure strategies) : (t, L) and (b, R)

but game 1 has no Nash equilibrium in pure strategies ; neither does this one, example 10

$$\begin{pmatrix} 1\backslash 2 & L & R \\ & & \\ t & (1,1) & (0,8) \\ b & (0,6) & (5,5) \end{pmatrix}$$

every solution by iterated elimination of weakly dominated strategies is also a pure strategy Nash equilibrium

but not vice versa

and there can be **other** Nash equilibria to games which can be solved by iterated elimination of weakly dominated strategies

(not true for iterated elimination of strictly dominated strategies)

example 5 has 2 Nash equilibria in pure strategies

$$\begin{pmatrix} 1\backslash 2 & L & R \\ & & \\ t & (5,5) & (0,0) \\ b & (0,0) & (0,0) \end{pmatrix}$$

(t, L) and (b, R)

even though (b, R) has each player playing a weakly dominated stratgy

Mixed Strategies

terminology revision:

what had been called just plain "strategies" $\rightarrow$ **pure strategies**

if player 1 has n pure strategies $(s_1, s_2, \ldots, s_n)$, then a **mixed strategy** σ is a vector (of dimension n) of probabilities with which she plays each pure strategy s_i

so σ is a mixed strategy for player 1 if

$$\sigma_i \geq 0 \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sigma_1 + \sigma_2 + \cdots + \sigma_n = 1 \tag{2}$$

each pure strategy is an example of a mixed strategy ; the pure strategy s_n is

$$\sigma = (0, 0, \ldots, 0, 1)$$

for example

so players are now allowed to randomize over their mixed strategies

if player 1 plays a mixed strategy σ and player 2 plays a mixed strategy τ, what are the players' payoffs?

Rule : the payoff to player 1, when she plays the mixed strategy σ, and when player 2 plays the mixed strategy τ is

$$Eu^1 = \sum_{i=1}^{n} \sum_{j=1}^{m} \sigma_i \tau_j u^1(s_i, t_j)$$

(when player 1's pure strategies are $\{s_1, s_2, \ldots, s_n\}$ and player 2's pure strategies are $\{t_1, t_2, \ldots, t_m\}$)

underlying the rule

1. players are expected utility maximizers

2. they are randomizing **independently** : probability of strategy pair (s_i, t_j) occurring is $\sigma_i \tau_j$

Nash equilibrium in mixed strategies

(σ^N, τ^N) is a Nash equilibrium in mixed strategies if

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_i^N\tau_j^N u^1(s_i,t_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_i\tau_j^N u^1(s_i,t_j) \quad \text{all} \quad \sigma \in \Sigma \tag{3}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_i^N\tau_j^N u^2(s_i,t_j) \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sigma_i^N\tau_j u^2(s_i,t_j) \quad \text{all} \quad \tau \in \mathcal{T} \tag{4}$$

where Σ, $\mathcal{T}$ are the sets of all the mixed strategies for each player

Theorem 7.2

(Nash) : any game with a finite number of players, in which each player has a finite number of pure strategies, has at least 1 Nash equilibrium in mixed strategies

note : no matter what mixed strategy τ player 2 is playing, there is **no** completely mixed strategy σ which will give a higher expected payoff than any of the pure strategies

either there is exactly 1 pure strategy which is player 1's best response, or several pure strategies are tied for player 1's best response — in which case any mixed strategy will also be a best response, if it involves picking (with positive probability) only those pure strategies which are best

example 10

$$\begin{pmatrix} 1\backslash 2 & L & R \\ & & \\ t & (1,1) & (0,8) \\ b & (0,6) & (5,5) \end{pmatrix}$$

when would player 1 be willing to mix?

only if t and b are tied

that is, only if

$$1 \cdot \tau_1 + 0 \cdot \tau_2 = 0 \cdot \tau_1 + 5 \cdot \tau_2 \tag{5}$$

since $\tau_2 = 1 - \tau_1$, condition (5) can be written

$$\tau_1 = 5(1 - \tau_1) \tag{6}$$

or $\tau_1 = \frac{5}{6}$

similarly, player 2 would be willing to mix between his strategies only if

$$\sigma_1 \cdot 1 + \sigma_2 \cdot 6 = \sigma_1 \cdot 8 + \sigma_2 \cdot 5 \tag{7}$$

or

$$\sigma = (\frac{1}{8}, \frac{7}{8})$$

that's the (only) mixed strategy Nash equilibrium to example 10 :

$$\sigma = (\frac{1}{8}, \frac{7}{8}); \tau = (\frac{5}{6}, \frac{1}{6})$$

because : only if player 2 plays $\tau = (\frac{5}{6}, \frac{1}{6})$ will player 1 be willing to randomize, and only if player 1 plays $\sigma = (\frac{1}{8}, \frac{7}{8})$ will player 2 be willing to randomize