Beliefs and Sequential Equilibrium

to solve a game of incomplete information, we should look at the **beliefs** of the uninformed player(s)

suppose that player 2 is in an information set which contains two nodes (such as the information set leading from player 1's choice of action a in game 6)

his beliefs are the probabilities that he assigns to the nodes in the information set

that is, in the example, his beliefs are the probabilities he assigns to the event "nature chose its 'top' action", **given** that he has just observed player 1 choose a

so part of solving the game is calculating the probability $P(top|a)$: the probability that nature chose "top", given that player 1 just chose a

Bayes's Rule

person 2's prior probability assigned to nature choosing "top" was $1/4$, in game 6

but he has acquired some information since then : player 1 has chosen a

we will require that player 2's posterior beliefs (after he has observed an action by player 1) be consistent with Bayes's Rule

$$P(top|a) = \frac{P(a|top)P(top)}{P(a|top)P(top) + P(a|bot)P(bot)} \quad (1)$$

(where top and bot are nature's two moves)

and $P(top)$ and $P(bot)$ are his prior beliefs : $1/4$ and $3/4$ in example 6 given a set of beliefs, player 2 can figure out his optimal strategy :

if player 1 just played a, then, in example 6, player 2's payoff from choosing the action B would be

$$2P(top|a) + 12P(bot|a) = 12 - 10P(top|a) \qquad (2)$$

and his payoff from choosing A would be $5P(top|a)$

required (so far)

i player 2's beliefs are consistent with Bayes's Rule

ii player 2's choice of action (at each node) maximizes his expected payoffs — given his beliefs so in example 6, he would choose the action A if and only if

$$P(top|a) \leq \frac{4}{5} \quad (3)$$

now where did the probabilities such as $P(a|top)$ used in Bayes's Rule come from?

$P(a|top)$ is the probability that player 1 chooses the action a, given that she just observed nature choose top

of course the $P(a|top)$ was used by player 2 in his calculation, in updating his beliefs ; it's his guess as to the probability with which player 1 would choose this action, given her observation of nature's move

in equilibrium, player 2's guesses have to be correct

that is

iii the probabilities $P(a|top)$ (etcetera) that player 2 uses in his Bayesian updating must be the actual probabilities with which player 1 chooses this action (a) at this node (top)

and what determines the probability with which player 1 chooses her actions?

she chooses her actions to maximize her expected payoff **given** the actions that she expects player 2 to take

in equilibrium, her guesses about player 2's strategies must be correct, so that we require

iv player 1 will choose action a with positive probability at the node top only if a maximizes her expected payoff, given player 2's choices of actions at subsequent nodes

Sequential Equilibrium

a set of mixed strategies for player 1, of mixed strategies by player 2, and of beliefs for player 2 will constitute a **sequential equilibrium** to the game only if they obey requirements $i - iv$

actually, to be a sequential equilibrium, a set of strategies and beliefs must also satisfy an additional "technical" requirement, outlined in definition 7.20 in *Jehle and Reny* ; an equilibrium satisfying (only) i – iv is a "perfect Bayesian equilibrium", which is a (very slightly) weaker equilibrium concept

Theorem 7.7: every finite game of incomplete information has at least one sequential equilibrium

Extensive Form Game 6

a sequential equilibrium :

player 1's strategy : if nature 's move was top, play b

: if nature's move was bot, play a

player 2's beliefs :

$$P(top|a) = 0.00$$

$$P(top|b) = 1.00$$

player 2's strategy :

always play B

why is this a sequential equilibrium?

Bayes's Rule : since 1 plays a (for certain) if and only if nature played bot, if player 2 sees a, he knows nature's move was bot, and if he sees b he knows nature's move was top

player 2's best response : given his beliefs

if 1 played a, then his payoff from A is 0 and from B is 12

if 1 played b his payoff from A is 2 and from B is 4

player 1's best response :

player 2 always plays B

so if nature's move was top, a gives her 8 and b gives her 12

if nature's move was bot, a gives her 12 and b gives her 3

that's it

Extensive Form Game 7

a sequential equilibrium :

player 1's strategy :

if nature plays top, play a with probability $8/9$, and b with probability $1/9$

if nature plays bot, play a for sure

player 2's beliefs

$$P(top|a) = \frac{8}{11}$$
$$P(top|b) = 1$$

player 2's strategy :

if 1 plays a, play A with probability $1/3$ and B with probability $2/3$

if 1 plays b, play A for sure

that's not the only sequential equilibrium

here's another

player 1's strategy : play a no matter what is nature's move

player 2's beliefs :

$$P(top|a) = \frac{3}{4}$$

$$P(top|b) = \frac{1}{5}$$

player 2's strategy

if 1 plays a, play A for sure

if 1 plays b, play B for sure

where did that $P(top|b)$ come from?

any beliefs about $P(top|b)$ are consistent with Bayes's Rule — since b never actually gets chosen by player 1 along the equilibrium path

"out of equilibrium beliefs"