What is an Equilibrium Price Vector?

it's a vector $\mathbf{p}$ such that $\mathbf{Z}(\mathbf{p}) = 0$

where $\mathbf{Z}(\mathbf{p})$ is a vector of excess demands

in general equilibrium, we are looking for prices that clear all n markets

there are n prices : $(p_1, p_2, p_3, \ldots, p_n)$

and n markets to clear

$$Z_1(\mathbf{p}) = 0 \quad , \quad Z_2(\mathbf{p}) = 0 \quad , \ldots , \quad Z_n(\mathbf{p}) = 0$$

so n equations in n unknowns

What was Good Enough for Walras ...

Walras concluded that an equilibrium price vector $\mathbf{p}$ must exist (given endowments,preferences)

but it's not in general true that there must exist a solution to n equations in n unknowns

there is no solution to the 2 equations in 2 unknowns

$$x^2 + y^2 = -4$$
$$x^2 + y^2 = 6$$

and even though there is a solution to the 2 equations in 2 unknowns

$$p_1 + p_2 = 0$$
$$p_1 - 3p_2 = 6$$

the solution has $p_2 = -1.5 < 0$, which makes no sense for a price

Many Solutions

since each person's Marshallian demand function is homogeneous of degree 0 in prices and income together

and since here income of person i is $p \cdot \mathbf{e}^i$ which is proportional to prices

therefore each excess demand function $\mathbf{z}^i(\mathbf{p})$ is homogeneous of degree 0 in prices

so if $\mathbf{Z}(\mathbf{p}) = 0$ for some price vector $\mathbf{p}$, then $\mathbf{Z}(a\mathbf{p}) = 0$ for any positive constant a

which means we really do not have n unknown prices : only $n - 1$

for example, we could normalize prices by making good 1 the numéraire and setting $p_1 \equiv 1$

Walras's Law (Again)

suppose that a price vector $\mathbf{p}$ clears the markets for goods $1, 2, 3, \ldots, n-1$

that is : $Z_j(\mathbf{p}) = 0$ for $j = 1, 2, \cdots, n-1$

Walras's Law says that $\sum_{j=1}^{n} p_j Z_j(\mathbf{p}) = 0$ for any price vector $\mathbf{p}$

so that

$$Z_n(\mathbf{p}) = -\frac{1}{p_n} \sum_{j=1}^{n-1} p_j Z_j(\mathbf{p})$$

if markets for goods $1, 2, \ldots, n-1$ clear, then the market for good n **must** clear, as a consequence of Walras's Law

so we only have $n-1$ independent equations defining Walrasian equilibrium

Fixed Point Theorems

the way that Arrow, Debreu and McKenzie demonstrated that, despite these complications, that a market–clearing price vector will exist, was to use a **fixed point theorem**

(Math) : if a function ϕ maps some set S into S, then some point $s \in S$ is called a fixed point for the function if $\phi(s) = s$

Brouwer's Fixed Point Theorem : if S is a **compact**, **convex** set, and if ϕ is a continuous function mapping S into itself, then ϕ must have a fixed point

relevance? suppose that we adjust the price whenever the market does not clear : raise p_j if $Z_j(\mathbf{p}) > 0$, and lower it if $Z_j(\mathbf{p}) < 0$

then if $\phi(\mathbf{p})$ is my new adjusted price, $\mathbf{p} = \phi(\mathbf{p})$ if and only if all markets clear

complications : what's the set S here?

use the normalization that the sum of all prices is 1

S : the set of all $(p_1, p_2, \cdots, p_n)$ such that each $p_j \geq 0$,and $p_1 + p_2 + \cdots + p_n = 1$

(Walras's Law lets me do that)

more complications : we have to ensure that the price stays in the set S after we adjust it

so we have to raise prices in markets with positive excess demand, and lower them in markets with negative excess demand, in such a way that the prices stay non–negative, and their sum stays at 1

for details : see *Jehle and Reny*