The Fundamental Theorems of Welfare Economics

start with a given exchange economy : I people, each with preferences (represented by some utility functions $u^i(\mathbf{x}^i)$), and each with endowment vector $\mathbf{e}^i$

definitions

a price vector $\mathbf{p}^*$ is a Walrasian equilibrium price vector for this economy if $\mathbf{Z}(\mathbf{p}^*) = 0$

an allocation $\mathbf{x}$ is a Walrasian equilibrium allocation for this economy if

$$\mathbf{x}^i = \mathbf{x}^{iM}(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) \quad (1)$$

for each person i

where $\mathbf{x}^{iM}(\mathbf{p}, y)$ is person i's Marshallian demand function,and $\mathbf{p}^*$ is a Walrasian equilibrium price vector

Theorem 5.6

if $\mathbf{x}$ is a Walrasian equilibrium allocation, then $\mathbf{x}$ is in the core

proof : uses revealed preference (remember?)

so if $\mathbf{x}$ is a Walrasian equilibrium allocation, we must show that no coalition S can block it with some other allocation $\mathbf{y}$

now

$$\mathbf{x}^i = \mathbf{x}^{iM}(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) \tag{2}$$

so that if $u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)$, then

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i \tag{3}$$

with strict inequality if $u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i)$

also

$$\mathbf{p}^* \cdot \mathbf{x}^i = \mathbf{p}^* \cdot \mathbf{e}^i \tag{4}$$

since $\mathbf{x}^i$ is on person i's budget line (at prices $\mathbf{p}^*$)

so suppose that everyone person in coalition S likes $\mathbf{y}^i$ at least as much as $\mathbf{x}^i$, and at least one person in S likes it strictly better

then (adding up equation (3) over everyone in S)

$$\sum_{i \in S} \mathbf{p}^* \cdot \mathbf{y}^i > \sum_{i \in S} \mathbf{p}^* \cdot \mathbf{x}^i \tag{5}$$

now use equation (4)

$$\sum_{i \in S} \mathbf{p}^* \cdot \mathbf{y}^i > \sum_{i \in S} \mathbf{p}^* \cdot \mathbf{e}^i \tag{6}$$

but if S blocks $\mathbf{x}$ with $\mathbf{y}$, it must be true that

$$\sum_{i \in S} \mathbf{y}^i \leq \sum_{i \in S} \mathbf{e}^i \tag{7}$$

equations (6) and (7) cannot both hold : they're inconsistent with each other

conclusion : no coalition can block a Walrasian equilibrium allocation

First Fundamental Theorem

recall : any core allocation must be Pareto efficient

so Theorem 5.6 implies immediately

First Fundamental Theorem of Welfare Economics : any Walrasian equilibrium allocation must be Pareto efficient

another implication of 5.6 :

there core of any exchange economy is non–empty

Second Fundamental Theorem

suppose that $\mathbf{x}$ is some Pareto efficient allocation : then there is some division $(\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^I)$ of the aggregate endowments among the people, so that $\mathbf{x}$ is a Walrasian equilibrium allocation for this endowment pattern

i.e. : any Pareto efficient allocation can be achieved as a competitive equilibrium, after some re–arrangement of people's endowments

proof :

start now with some Pareto efficient allocation $\mathbf{x}$

we need to find prices and endowments to make this into a Walrasian equilibrium allocation

prices?

let $p_1^* = 1$ (just choosing a numéraire)

now let

$$p_j^* = \frac{u_j^1(\mathbf{x}^1)}{u_1^1(\mathbf{x}^1)} \tag{8}$$

since $\mathbf{x}$ is Pareto efficient, $\frac{u_j^i}{u_1^i} = \frac{u_j^1}{u_1^1}$ for each person i, so that

$$\frac{u_j^i(\mathbf{x}^i)}{u_k^i(\mathbf{x}^i)} = \frac{p_j^*}{p_k^*} \tag{9}$$

for each person i, and each pair of goods j, k

that means that person i would demand the consumption bundle $\mathbf{x}^i$, if she faced prices $\mathbf{p}^*$

provided that her income were “right”: provided her income equalled $\mathbf{p}^* \cdot \mathbf{x}^i$

so just pick $\mathbf{e}^i$ as any vector on the price line through $\mathbf{x}^i$: any $\mathbf{e}^i$ such that

$$\mathbf{p}^* \cdot \mathbf{e}^i = \mathbf{p}^* \cdot \mathbf{x}^i \tag{10}$$

we always can find such endowments —

$\mathbf{e}^i \equiv \mathbf{x}^i$ for example

so a price vector $\mathbf{p}^*$, and a division $(\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^I)$ of the aggregate endowments have been found, which make the given Pareto efficient allocation $\mathbf{x}$ a Walrasian equilibrium allocation for the economy