A Simple Portfolio Allocation Problem

W_0 : initial wealth

X : investment in risky asset

r_s : (certain) return on safe asset

r : (uncertain) return on risky asset

$W = (1+r)X + (1+r_s)(W_0 - X)$: end–of–period wealth

choose X to maximize EU

$$EU = E(u[(1+r)X + (1+r_s)(W_0 - X)]) \tag{1}$$

$$EU = \int u[(1+r)X + (1+r_s)(W_0 - X)]f(r)dr \quad (2)$$

first–order condition

$$E(u'[(1+r)X + (1+r_s)(W_0 - X)][r - r_s]) = 0 \quad (3)$$

or

$$E(u'(W)(r - r_s)) = 0 \quad (4)$$

second–order condition

$$\frac{\partial^2 EU}{\partial X^2} = E(u''(W)((r - r_s)^2) < 0 \quad (5)$$

$X > 0$ **?**

at $X = 0$,

$$\frac{\partial EU}{\partial X} = u'(W_0[1+r_s])E(r-r_s) = u'(W_0[1+r_s])(E(r)-r_s) \qquad (6)$$

so that the optimal $X > 0$ whenever $E(r) > r_s$

How Does X Vary with W_0?

differentiate first–order condition (4)

$$(1+r_s)E[u''(W)(r-r_s)]dW_0+E[u''(W)(r-r_s)^2]dX = 0 \tag{7}$$

so that

$$\frac{\partial X}{\partial W_0} = -(1+r_s)\frac{E[u''(W)(r-r_s)]}{E[u''(W)(r-r_s)^2]} \tag{8}$$

denominator < 0 (2nd–order condition (5))

numerator is

$$-(1+r_s)E[\frac{u''(W)}{u'(W)}U'(W)(r-r_s)] \tag{9}$$

which equals

$$= (1 + r_s)E[R_a(W)u'(W)(r - r_s)] \qquad (10)$$

let W_1 be end–of–period wealth if $r = r_s$

$$W_1 \equiv (1 + r_s)W_0$$

then expression (10) can be written as $A+B+C$ with

$$A = (1 + r_s)R_a(W_1)E[u'(W)(r - r_s)$$

$$B = (1+r_s)\int_{-1}^{r_s}(R_a(W)-R_a(W_1))u'(W)(r-r_s)f(r)dr$$

$$C = (1+r_s)\int_{r_s}^{\infty}(R_a(W)-R_a(W_1))u'(W)(r-r_s)f(r)dr$$

$A = 0$ from the first–order condition (3)

if $\frac{\partial R_a}{\partial W} < 0$, then whenever $r < r_s$, so that $W < W_1$, $R_a(W) > R_a(W_1)$, so that expression B is negative

similarly, $DARA$ implies that $R_a(W) < R_a(W_1)$ whenever $r > r_s$ and $W > W_1$, meaning that term C is negative

therefore, $DARA$ implies that $A + B + C < 0$

since the denominator in expression (8) is negative, from the second–order conditions, therefore

with $DARA$, $\frac{\partial X}{\partial W_0} > 0$