GS/ECON 5010 Answers to Assignment 1 October 2004

1. (a) $u(x_1, x_2) = x_1 + x_1 x_2$

This is a differentiable function, and the partial derivatives are

$$\frac{\partial u}{\partial x_1} = 1 + x_2$$

$$\frac{\partial u}{\partial x_2} = x_1$$

so that the partial derivatives are positive whenever $\mathbf{x} >> 0$ (and non–negative whenever $\mathbf{x} \geq 0$). Also, $u(x_1, x_2) > 0$ whenever $x_1 > 0$, and $u(0, x_2) = 0$, so that $u(x_1, x_2) > u(0, 0)$ whenever $(x_1, x_2) >> 0$. So preferences are stictly monotonic.

One way of checking convexity of preferences is to look at the shape of the indifference curves. The equation of an indifference curve is

$$x_1 + x_1 x_2 = A$$

for any constant A, or

$$x_2 = \frac{A - x_1}{x_1} = \frac{A}{x_1} - 1$$

That equation defines a curve which gets less steep as we move down and to the right : the slope is $-A/((x_1)^2)$: so that preferences are convex.

Alternatively, we could check that $\delta' M \delta \leq 0$ for any direction vector δ such that $\nabla u \cdot \delta = 0$, where M is the matrix of second derivatives, and ∇u is the vector of first derivatives. Here

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that $\delta' M \delta = 2\delta_1 \delta_2$. And $\nabla u \cdot \delta = \delta_1(1 + x_2) + \delta_2 x_1$, so that $\nabla u \cdot \delta = 0$ if and only if $\delta_2 = -\delta_1 \frac{1+x_2}{x_1}$, implying that

$$\delta' M \delta = -2(\delta_1)^2 \frac{1 + x_2}{x_1} < 0$$

showing that the utility function is quasi–concave, and hence preferences are convex.

(b) $u(\mathbf{x}) = b\mathbf{x} + \mathbf{x}' A \mathbf{x}$ where b is a vector of positive numbers, and A is a matrix with positive numbers on the diagonal, and zeroes off the diagonal

Here the partial derivative of utility with respect to consumption of good i is

$$\frac{\partial u}{\partial x_i} = b_i + A_{ii} x_i > 0$$

so that preferences are strictly monotonic.

But this utility function is strictly convex : the matrix of second derivatives is A, which is a positive definite matrix.

If a utility function is convex, then the preferences it represents cannot be strictly convex.

Why not? If $u(\mathbf{x})$ is a strictly convex function, then

$$u(\alpha \mathbf{x}^1 + (1-\alpha)\mathbf{x}^2) < \alpha u(\mathbf{x}^1) + (1-\alpha)u(\mathbf{x}^2) \qquad (1-1)$$

for any distinct consumption bundles $\mathbf{x}^1$ and $\mathbf{x}^2$, whenever $0 < \alpha < 1$.

Now suppose that $u(\mathbf{x^1}) = \mathbf{u}(\mathbf{x^2})$. That means that both $\mathbf{x}^1$ and $\mathbf{x}^2$ are in the set of consumption bundles which are weakly preferred to $\mathbf{x}^1$. But equation $(1-1)$ says that $\alpha \mathbf{x}^1 + (1-\alpha)\mathbf{x}^2$ is not weakly preferred to $\mathbf{x}^1$. Therefore the weakly preferred set to $\mathbf{x}^1$ is not convex, so that preferences are not convex.

Alternatively, the definition above of the partial derivatives shows that the (absolute value of the) slope of an indifference curve in x_1–x_2 space is

$$MRS = \frac{b_1 + A_{11}x_1}{b_2 + A_{22}x_2}$$

As we move down the curve, and to the right (with x_1 on the horizontal axis), this slope gets bigger, not smaller, showing that the indifference curves are not convex to the origin in this case.

2. This problem is probably best done using indifference curves.

Given these preferences, the indifference curves are straight lines with slope $-1/2$, if $x_1 > x_2$, and straight lines with slope -2 if $x_2 > x_1$. There is a kink in each indifference curve, at the point $x_1 = x_2$: $2x_1 + x_2 > x_1 + 2x_2$ if and only if $x_1 > x_2$.

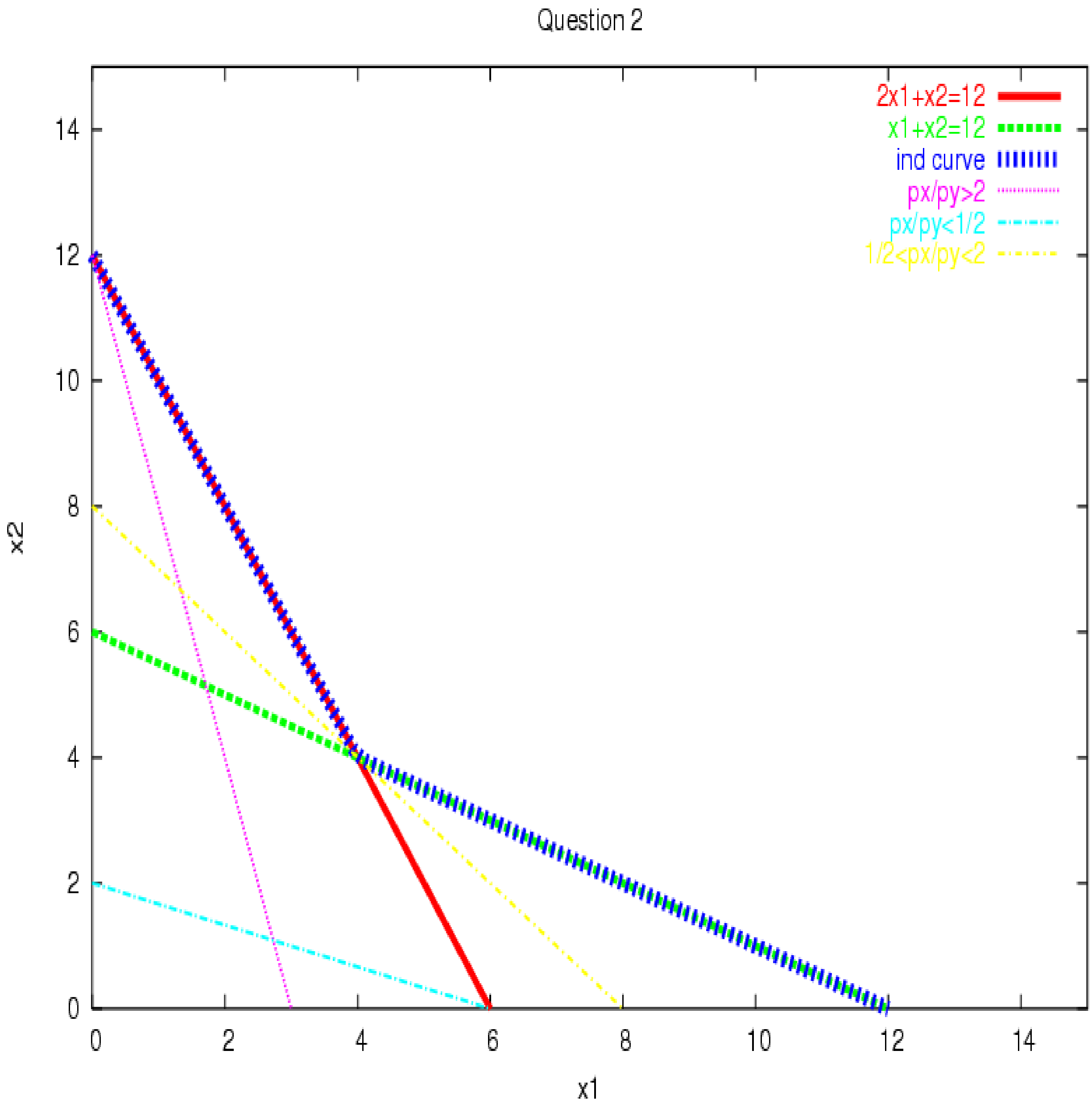
These preferences are not strictly convex ; the indifference curves consist of line segments.

There won't be neat tangencies of indifference curves with budget lines, but instead there may be corner solutions.

If $p_1 > 2p_2$, then the person will want to spend all her income on commodity #2. If she does spend all her money on commodity #2, then she has $x_1 = 0$, $x_2 = y/p_2$, and a utility level of $2 \cdot 0 + y/p_2 = y/p_2$. If she were to move from the consumption bundle $(0, y/p_1)$, along her budget line, then she could decrease x_2 by some ϵ, and increase x_1 by $\frac{p_2}{p_1}\epsilon$ (since she is spending $p_2\epsilon$ less on good 2, and can spend that money on good 1). Since $u = 2x_1 + x_2$ (when $x_1 < x_2$), her utility would increase by $2\frac{p_2}{p_1}\epsilon - \epsilon$, which is negative if $p_1 > 2p_1$.

So when $p_1 > 2p_2$ she chooses the consumption bundle $(0, y/p_2)$; when $2p_2 > p_1 > p_2/2$, she chooses to consume at the kink on her indifference curve, where $x_1 = x_2$, and so chooses the consumption bundle $(y/(p_1+p_2), y/(p_1+p_2))$; if $p_2 > 2p_1$ she chooses the other corner solution, $(y/p_1, 0)$. If $p_1 = 2p_2$, then she is indifferent among all bundles along the line segment connecting $(0, y/p_2)$ with $(y/(p_1+p_2), y/(p_1+p_2))$, and if $p_2 = 2p_1$, then she is indifferent among all bundles along the line segment connecting $(y/(p_1+p_2), y/(p_1+p_2))$ with $(y/p_1, 0)$.

Solving this problem using caculus should also suggest what the answer is. For example, if $p_1 > 2p_2$, then there will be no soultion to the problem of maximizing $2x_1 + x_2$ subject to the constraint that $p_1x_1 + p_2x_2 \leq y$; increasing x_2 will always increase the value of utility, which shows that the consumer will want to increase x_2 as much as she can — up to the point at which $x_1 = 0$.


Question 2
2x1+x2=12
x1+x2=12
ind curve
px/py>2
px/py<1/2
1/2<px/py<2
x1
x2
0
2
4
6
8
10
12
14

3. If

$$u(\mathbf{x}) \equiv (a_1 x_1^{\rho} + a_2 x_2^{\rho} + \cdots + a_n x_n^{\rho})^{1/\rho}$$

then the marginal utility of consumption of good i is

$$\frac{\partial u}{\partial x_i} = a_i [\sum_{j=1}^{n} a_j x_j^{\rho}]^{1/\rho - 1} (x_i)^{\rho - 1} \qquad i = 1, 2, \ldots, n$$

implying that the marginal rate of substitution between goods i and j is

$$MRS_{ij} = \frac{a_i}{a_j} [\frac{x_i}{x_j}]^{\rho - 1}$$

so that the first order conditions for utility maximization are

$$\frac{x_i}{x_j} = [\frac{a_i}{a_j}]^{1/(1-\rho)} [\frac{p_i}{p_j}]^{1/(\rho - 1)}$$

for any two goods i or j, implying as well that

$$x_i = [\frac{a_i}{a_1}]^{1/(1-\rho)} [\frac{p_i}{p_1}]^{1/(\rho-1)} x_1 \qquad i = 1, 2, \ldots, n \tag{1}$$

Substituting into the budget constraint $\sum_{i=1}^{n} p_i x_i = y$ yields

$$[\sum_{i=1}^{n} p_i [\frac{a_i}{a_1}]^{1/(1-\rho)} [\frac{p_i}{p_1}]^{1/(\rho-1)}] x_1 = y$$

or

$$[\sum_{i=1}^{n} p_i^{\rho/(\rho-1)} a_i^{1/(1-\rho)}] x_1 = p_1^{\rho/(\rho-1)} a_1^{1/(\rho-1)} y \tag{2}$$

which implies that

$$x_1 = \frac{a_1^{1-r}}{\sum_{i=1}^{n} a_i^{1-r}} \frac{p_1^{r-1} y}{\sum_{i=1}^{n} p_i^{r}} \tag{3}$$

where

$$r \equiv \frac{\rho}{\rho - 1}$$

Substituting from equation (3) into equation (1) gives the Marshallian demand functions

$$x_j^M(\mathbf{p}, y) = \frac{a_j^{1-r}}{\sum_{i=1}^{n} a_i^{1-r}} \frac{p_j^{r-1} y}{\sum_{i=1}^{n} p_i^{r}} \qquad j = 1, 2, \ldots n \tag{4}$$

Note that the case in the textbook ($a_1 = a_2 = a_3 = \cdots = a_n$), and the Cobb–Douglas case $\rho = r = 0$ are special cases of equation (4).

4. If a person's preferences can be represented by the utility function

$$u(x_1, x_2) = x_1 + 2\sqrt{x_2}$$

then her marginal utilities of consumption of the 2 commodities are

$$\frac{\partial u}{\partial x_1} = 1$$

$$\frac{\partial u}{\partial x_2} = \frac{1}{\sqrt{x_2}}$$

so that her first–order condition for utility maximization is

$$MRS = u_1/u_2 = \sqrt{x_2} = \frac{p_1}{p_2} \tag{5}$$

Equation (5) yields — immediately — the Marshallian demand function for good 2,

$$x_2^M(\mathbf{p}, y) = [\frac{p_1}{p_2}]^2 \tag{6}$$

Substituting from equation (6) into the budget constraint yields

$$x_1 = \frac{y - p_2 x_2}{p_1} = \frac{y - p_2(p_1/p_2)^2}{p_1}$$

implying a Marshallian demand function

$$x_1^M(\mathbf{p}, y) = \frac{y}{p_1} - \frac{p_1}{p_2} \tag{7}$$

(Note : equations (6) and (7) are the Marshallian demand functions if and only if the demand defined by equation (7) is non–negative. If $y < (p_1)^2/p_2$, so that the right hand side of (7) is negative, then the consumer will be at a corner solution. She will choose $x_1^M(\mathbf{p}, y) = 0$, and $x_2^M(\mathbf{p}, y) = y/p_2$.)

Substituting from equations (6) and (7) into the consumer's direct utility function $u = x_1 + 2\sqrt{x_2}$ implies that her utility is

$$\frac{y}{p_1} - \frac{p_1}{p_2} + 2\sqrt{[p_1/p_2]^2}$$

so that her indirect utility function is

$$v(\mathbf{p}, y) = \frac{y}{p_1} + \frac{p_1}{p_2} \tag{8}$$

You can check, from differentiating equation (8) with respect to y, p_1 and p_2, and comparing the answer to equations (6) and (7), that

$$x_i^M(\mathbf{p}, y) = -\frac{v_i(\mathbf{p}, y)}{v_y(\mathbf{p}, y)} \qquad i = 1, 2$$

in accordance with Roy's Identity.

It is easiest to proceed next by finding the expenditure function. The relation between the expenditure function and the indirect utility function says that

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

substituting from equation (8) implies that

$$\frac{e(\mathbf{p}, u)}{p_1} + \frac{p_1}{p_2} = u$$

which then implies

$$e(\mathbf{p}, u) = p_1 u - \frac{(p_1)^2}{p_2} \tag{9}$$

Now that the expenditure function has been derived, the Hicksian demand functions are simply the derivatives of that expenditure function with respect to the prices.

$$x_1^H(\mathbf{p}, u) = e_1(\mathbf{p}, u) = u - 2\frac{p_1}{p_2} \tag{10}$$

$$x_2^H(\mathbf{p}, u) = e_2(\mathbf{p}, u) = [\frac{p_1}{p_2}]^2 \tag{11}$$

You can check that equations (6), (7), (10) and (11) satisfy the Slutsky equation.

Note that here, because preferences are **quasi–linear**, the Marshallian demand function for good #2 is independent of the consumer's income. because there is no income effect (on demand for good #2), here the Hicksian and Marshallian demand functions for good 2 are identical.