GS/ECON 5010 Answers to Assignment 4 November 2004

Q1. What does the contract curve look like for a 2–person, 2–good exchange economy, if each person has the same preferences, represented by the utility function

$$u(x_1^i, x_2^i) = \ln x_2^i - \frac{1}{x_1^i}$$

where x_j^i is person i's consumption of good j?

A1. Here each person's marginal rate of substitution is

$$MRS^i = \frac{(1/(x_1^i)^2)}{(1/x_2^i)} = \frac{x_2^i}{(x_1^i)^2}$$

so that the equation of the contract curve is

$$\frac{x_2^1}{(x_1^1)^2} = \frac{e_2 - x_2^1}{(e_1 - x_1^1)^2} \tag{1-1}$$

Equation $(1-1)$ can be written

$$x_2^1(e_1 - x_1^1)^2 = (x_1^1)^2(e_2 - x_2^1) \tag{1-2}$$

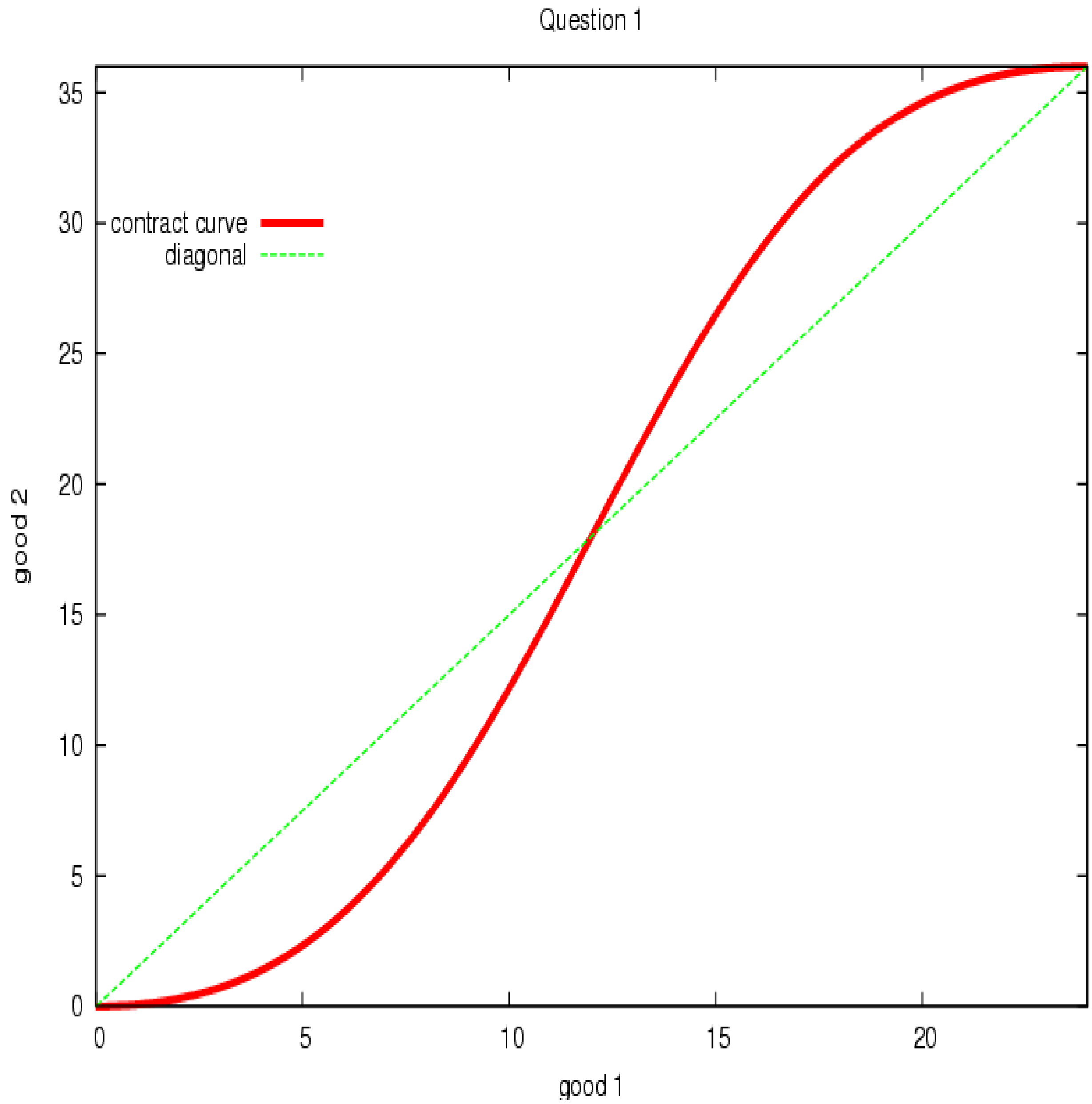
or

$$x_2^1 = \frac{(x_1^1)^2 e_2}{(e_1)^2 - 2x_1^1 e_1 + 2(x_1^1)^2} \tag{1-3}$$

This is an upward–sloping curve, going through both corners of the Edgeworth box. Differentiating $(1-3)$, the slope of the contract curve is

$$\frac{\partial x_2^1}{\partial x_1^1} = \frac{2x_1^1 e_2}{(e_1)^2 - 2x_1^1 e_1 + 2(x_1^1)} + 2(e_1 - 2x_1^1)\frac{x_2^1}{(e_1)^2 - 2x_1^1 e - 1 + 2(x_1^1)^2} \tag{1-4}$$

This slope is 0, at both corners $(0,0)$ and (e_1, e_2) of the Edgeworth box. As the figure shows, the contract curve is S–shaped : a person here has a stronger taste for good 1 when she is poor than when she is rich.


Question 1
contract curve
diagonal
good 2
good 1
0
5
10
15
20
25
30
35
0
5
10
15
20

Q2. What would the contract curve be in a 2–person, 2–good exchange economy, in which person 1 actually cared about person 2's well–being? That is, person 1's utility depends on her own consumption, but also on the consumption of person 2. In particular, let person 1's preferences be represented by the utility function

$$u^1(\mathbf{x}) = 10x_1^1 + 2\sqrt{x_1^2 x_2^2}$$

and person 2's by the utility function

$$u^2(\mathbf{x}) = 2\sqrt{x_1^2 x_2^2}$$

where x_j^i is person i's consumption of good j. (So here person 1 does not care about her own consumption of good #2, and person 2 has "standard" preferences, not caring about person 1's consumption.) Let the total endowment $\bar{e}$ of each good be $e_1 = 5$ and $e_2 = 200$.

A2. First of all, in this case efficiency dictates giving all of good 2 to person 2. Person 1 gets no direct benefit from consumption of good 2, and gets some (indirect) benefit from person 2 consuming good 2. So any Pareto efficient allocation here will have $x_2^2 = e_2$ and $x_2^1 = 0$.

That leaves the allocation of good 1. Since

$$x_1^1 = e_1 - x_1^2$$

person 1's utility can be written

$$10(e_1 - x_1^2) + 2\sqrt{x_1^2 e_2}$$

The derivative of this utility with respect to x_1^2 is

$$\sqrt{\frac{e_2}{x_1^2}} - 10$$

This derivative will be positive if and only if

$$\frac{e_2}{x_1^2} > 100$$

or

$$x_1^2 < \frac{e_2}{100}$$

If u^1 is increasing in x_1^2, then the allocation is not efficient : increasing x_1^2 will make person 2 better off, but it will also make person 1 better off, since the indirect increase in her utility (through her caring about the other person) is bigger than the direct decrease caused by x_1^1 falling.

On the other hand, if $x_1^2 > \frac{e_2}{100}$, further increases in x_1^2 will make person 2 better off but person 2 worse off. there is no way of making both people better off, so that the allocation is Pareto efficient.

Given that $e_1 = 5$ and $e_2 = 200$ here, the Pareto efficient allocations are any allocations with $x_2^2 = 200$, and $x_1^2 \geq 2$.

Q3. What allocations would be in the core of an exchange economy containing 2 million people, 1 million of whom liked only good 1, and 1 million of whom liked only good 2, if each person were endowed with 1 unit of each good? Explain briefly.

A3. The "obvious" core allocation here is to give each type–1 person 2 units of good 1, and each type–2 person 2 units of good 2 (where the "type–i" people are the people who like only good i).

That is not the only Pareto efficient allocation. Allocations do not have to treat each of the 1 million type–1 people identically. So another Pareto efficient allocation would be to give all the 2,000,000 units of good 1 to the first type–1 person, nothing to the remaining 999,999 type–1 people, and 1 unit each of good 2 to each of the type–2 people. In fact, any allocation will be Pareto efficient if it divides the available stock of good 1 (in any way at all) among the 1 million type–1 people, and the available stock of good 2 among the type–2 people.

But it turns out that allocations which do not treat identical people identically will **not** be in the core.

Suppose that some Pareto efficient allocation gave some type–1 person less than 2 units of good 1. Call this person "Bert". Now there must be a type–2 person who gets 2 or fewer units of good 2 (since there are 2 million units of good 2 in total, and 1 million type–2 people). Call this second person "Ernie". Now Bert and Ernie could form a coalition on their own. This coalition has a total endowment of 2 units of each good. So a feasible allocation for this two–person coalition is to give Bert 2 units of good 1, and Ernie 2 units of good 2. That means the coalition can block any allocation for which Bert gets less than 2 units of good 1 (and for which Ernie gets no more than 2 units of good 2).

So the above example, with a blocking coalition of one person of each type, shows that any core allocation must treat all people of a given type identically. The only Pareto efficient allocation which treats all people the same is the allocation in which each type–i person gets 2 units of good i.

[Since the core has to have at least one allocation, the "obvious" core allocation is indeed in the core. But it can be shown directly not to be blocked : if a coalition has M type–1 people in in, and $H < M$ type–2 members, then that coalition must give one of the type–1 people an allocation of $(H+M)/H < 2$ units of good 1, so that it cannot block the "obvious" core allocation. Similarly, a coalition with H type–1 members and $H > M$ type–2 members must make one of the type–2 members worse off. And a coalition with M members of each type certainly cannot do strictly better for all its members without making one of them worse off. So no coalition can block the allocation which gives each type–i person an allocation of 2 units of good i.]

Q4. How would the equilibrium prices of the goods vary with the people's endowments in a 2–person, 2–good exchange economy, if person 1 had preferences represented by the utility function

$$u^1((\mathbf{x}^1) = x_1^1 + \ln x_2^1$$

and person 2 had preferences represented by the utility function

$$u^2(\mathbf{x}^2) = x_1^2 x_2^2$$

where x_j^i is person i's consumption of good j?

A4. Since person 1 has $MU_1 = 1$, and $MU_2 = 1/x_2^1$, her Marshallian demand function for good 2 is

$$\frac{x_2^{1M}(\mathbf{p}, y^1) = p_1}{p_2} \tag{4 − 1}$$

Person 2 has Cobb–Douglas preferences, so his Marshallian demand function for good 2 is

$$x_2^{2M}(\mathbf{p}, y^2) = \frac{y_2}{2p_2} \tag{4 − 2}$$

Since person 2's income is the value $p_1 e_1^2 + p_2 e_2^2$ of his endowment, his total demand for good 2 is

$$x_2^2 = \frac{1}{2}[\frac{p_1}{p_2} e_1^2 + e_2^2] \tag{4 − 3}$$

Now each person's excess demand function for good 2 is just her or his total demand for the good, minus her or his endowment.

Since only relatively prices matter, the price of good 2 can be taken as numéraire, so that demands can be expressed in terms of the relative price

$$p \equiv \frac{p_1}{p_2}$$

Then equations $(4-1)$ and $(4-3)$ imply that

$$z_2^1(p) = p - e_2^1 \tag{4 − 4}$$

$$z_2^2(p) = \frac{1}{2}(pe_1^2 - e_2^2) \tag{4 − 5}$$

where z_j^i is person i's excess demand for good j. Note that equations $(4-4)$ and $(4-5)$ imply that each person's excess demand for good 2 is, in this example, an increasing function of the relative price p of good 1, so that there will be a unique equilibrium price p which clears the market for good 2, for a given pattern of endowments.

The total excess demand for good 2 is

$$z_2(p) = p[1 + \frac{1}{2}e_1^2] - [e_2^1 + \frac{1}{2}e_2^2] \tag{4 − 6}$$

so that excess demand for good 2 is 0 if and only if

$$p = \frac{2e_2^1 + e_2^2}{2 + e_1^2} \tag{4 − 7}$$

Equation $(4-7)$ defines the price p for good 1which clears the market for good 2, when $p_2 = 1$. But Walras's law implies that the market for good 1 will also clear if and only if $(4-7)$ holds. And the homogeneity of degree zero of demands in all prices also implies that any price pair $p_1 = pa$, $p_2 = a$, for $a > 0$ will clear both markets.

Q5. What are the Pareto efficient allocations, and the competitive equilibrium allocations, for the following economy with externalities (so that the two fundamental theorems of welfare economics do not apply)? The are two people, 2 consumption goods, and one input. Each consumption good can be produced using the input : 1 unit of the input can produce 1 unit of good 1, or 1unit of good 2. Person i is endowed with z^i units of the input, where $z^1 + z^2 = 36$. Good 1 is food, and good 2 is cigarettes. Person 2 is a smoker, and person 1 is a non–smoker who is harmed by second–hand smoke. Person 1's utility function can be written

$$u^1(\mathbf{x}) = x_1^1 - (x_2^2)^2$$

and person 2's utility function can be written

$$u^2(\mathbf{x}) = x_1^2 + 5x_2^2 - (x_2^2)^2$$

A5. Since person 1 does not like smoke, she will never choose to consume any cigarettes. So we can set $x_2^1 = 0$, whether we are looking at Pareto efficient allocations, or competitive equilibrium allocations.

Next, since one unit of the input can be made into one unit of food, or one unit of cigarettes, then an allocation (x_1^1, x_1^2, x_2^2) will be feasible if and only

$$x_1^1 + x_1^2 + x_2^2 \leq z_1 + z_2 \tag{5 − 1}$$

To find the Pareto efficient allocations, consider maximization of person 1's utility, subject to person 2 getting a given level $\bar{U^2}$ of utility, and subject to the feasibility constraint $(5-1)$. The Lagrangean for this problem can be written

$$x_1^1 - (x_2^2)^2 + \lambda[x_1^2 + 5x_2^2 - (x_2^2)^2 - \bar{U^2}] + \mu[z_1 + z_2 - x_1^1 - x_1^2 - x_2^2] \tag{5 − 2}$$

Maximizing expression $(5-2)$ with respect to x_1^1, $x^2 - 1$ and x_2^2 yields the first–order conditions

$$1 - \mu = 0 \tag{5 − 3}$$

$$\lambda - \mu = 0 \tag{5 − 4}$$

$$5\lambda - \mu - 2(1+\lambda)x_2^2 = 0 \tag{5-5}$$

Substituting for $\lambda = \mu = 1$ in equation $(5-5)$

$$x_2^2 = 1 \tag{5-6}$$

is the unique efficient level of cigarette production (and of cigarette consumption by person 2).

So the Pareto efficient allocations are all allocations $\mathbf{x}$ with $x_2^1 = 0$, $x_2^2 = 1$, $x_1^1 \geq 0$, $x_1^2 \geq 0$, with $x_1^1 + x_1^2 = z_1 + z_2 - 1$. (There are additional Pareto efficient allocations if the level of utility given to person is less than 4, since we cannot have $u^2 < 4$ if $z_1^2 \geq 0$ and $z_2^2 = 1$. These other efficient allocations have $x_2^2 < 1$, $x_1^2 = 0$, $x_2^1 = 0$, and $x_1^1 = z_1 + z_2 - z_2^2$.)

Under perfect competition, the price of each consumption good must be the same as the price of the input, since each consumption good is made from exactly one unit of the input. So we must have $p_1 = p_2 = w$, if w is the input price, in any competitive equilibrium. It is most convenient to normalize by setting $p_1 = p_2 = w = 1$.

Person 1 does not smoke, so that she spends all her income on food. That means that her total demand for food can be written

$$x_1^1 = z_1 \tag{5-7}$$

Person 2 has $MU_1^2 = 1$, and $MU_2^2 = 5 - 2x_2^2$, so that her quantity demanded of cigarettes is

$$x_2^2 = 2 \tag{5-8}$$

since the prices of food and cigarettes are 1, so that $MU_2^2 = 1$ at her optimum. (Note that her demand for cigarettes is independent of her income, because of her quasi–linear preferences.) Her quantity demanded of food is whatever income she has left over after buying cigarettes, so that

$$x_1^2 = z_2 - 2 \tag{5-9}$$

(If her endowment of the input is so low that $z_2 < 2$, then she will choose $x_2^2 = z_2$, and $z_1^2 = 0$.) Note that the equilibrium allocation is feasible ; equations $(5-7)$–$(5-9)$ imply that $x_1^1 + x_1^2 + x_2^2 = z_1 + z_2$. The equilibrium allocation is inefficient if $z_2 > 2$, since there $x_2^2 = 2$, and efficiency requires less smoking, $x_2^2 = 1$, if $x_1^2 > 0$.

(If $z_2 \leq 1$, then the "corner solution" competitive equilibrium actually is efficient : it has $x_1^1 = z_1 > z_1 + z_2 - 1$, $x_2^1 = x_1^2 = 0$, and $x_2^2 \leq 1$.)