$Q1$. Could the following 3 equations be Hicksian demand functions (if the reference level of utility u were high enough so that $u + \ln p_2 + \ln p_3 \geq 2 + 2\ln p_1$)? Explain briefly.

$$x_1(\mathbf{p}, u) = u - 2 - 2\ln p_1 + \ln p_2 + \ln p_3$$
$$x_2(\mathbf{p}, u) = u + \frac{p_1}{p_2}$$
$$x_3(\mathbf{p}, u) = u + \frac{p_1}{p_3}$$

$A1$ What would the expenditure function $e(\mathbf{p}, u)$ be, if these three functions were the Hicksian demand functions? Since

$$e(\mathbf{p}, u) = p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u) + p_3 x_3^H(\mathbf{p}, u)$$

it would have to be the case that

$$e(\mathbf{p}, u) = (p_1 + p_2 + p_3)u + p_1(\ln p_2 + \ln p_3 - 2\ln p_1) \qquad (1-1)$$

if the Hicksian demand functions $x_i^H(\mathbf{p}, u)$ were the ones listed in the question.

The expenditure function $e(\mathbf{p}, u)$ defined in $(1-1)$ is homogeneous of degree 1 in prices : since $\ln(ka) = \ln k + \ln a$,

$$e(k\mathbf{p}, u) = k(p_1 + p_2 + p_3)u + kp_1(\ln p_2 + \ln k + \ln p_3 + \ln k - 2\ln p_1 - 2\ln k) = ke(\mathbf{p}, u) \qquad (1-2)$$

The derivatives of the expenditure function $(1-1)$ are

$$e_1(\mathbf{p}, u) = u + \ln p_2 + \ln p_3 - 2\ln p_1 - 2 = x_1^H(\mathbf{p}, u) \qquad (1-3)$$
$$e_2(\mathbf{p}, u) = u + \frac{p_1}{p_2} = x_2^H(\mathbf{p}, u) \qquad (1-4)$$
$$e_3(\mathbf{p}, u) = u + \frac{p_1}{p_3} = x_3^H(\mathbf{p}, u) \qquad (1-5)$$

so that Shephard's Lemma holds.

If a function (such as $e(\mathbf{p}, u)$ defined above) is homogeneous of degree t, then all its first derivatives must be homogeneous of degree $t-1$. So equation $(1-2)$ establishes that all of the functions $x_i(\mathbf{p}, u)$ are homogeneous of degree 0.

[This fact can be established more directly : $tp_i/tp_j = p_i/p_j$ so that $x_2(\mathbf{p}, u)$ and $x_3(\mathbf{p}, u)$ are homogeneous of degree 0 in prices ; $\ln tp = \ln t + \ln p$ so that $x_1(\mathbf{p}, u)$ is homogeneous of degree 0.]

The matrix H of second derivatives of $e(\mathbf{p}, u)$ is

$$H(\mathbf{p}, u) = \begin{pmatrix} -\frac{2}{p_1} & \frac{1}{p_2} & \frac{1}{p_3} \\ \frac{1}{p_2} & -\frac{p_1}{(p_2)^2} & 0 \\ \frac{1}{p_3} & 0 & -\frac{p_1}{(p_3)^2} \end{pmatrix}$$

This matrix is symmetric. [This must be the case if $x_i(\mathbf{p}, u)$ is the i–th derivative of some function $e(\mathbf{p}, u)$ as has already been shown.]

The principal minors [the determinants of the 1–by–1, 2–by–2 and 3–by–3 matrices in the top left corner of H] are

$$M_1 = -\frac{2}{p_1} < 0$$

$$M_2 = \frac{2}{(p_2)^2} - \frac{1}{(p_2)^2} = \frac{1}{(p_2)^2} > 0$$

$$M_3 = -2\frac{p_1}{(p_2)^2(p_3)^2} + \frac{p_1}{(p_2)^2(p_3)^2} + \frac{p_1}{(p_2)^2(p_3)^2} = 0$$

so that the matrix H of second derivatives of $e(\mathbf{p}, u)$ is negative semi–definite, which means that the expenditure function $e(\mathbf{p}, u)$ defined by $(1-1)$ is a concave function.

The system of functions $\mathbf{x}(\mathbf{p}, u)$ are Hicksian demand functions if (and only if) they : (i) are all homogeneous of degree 0 in prices and (ii) have a matrix of derivatives which is symmetric and negative semi–definite, so that the functions defined in the question could be a system of Hicksian demand functions.

[The matrix H must be negative **semi**–definite, and not negative definite : its determinant must be 0 if the $x_i(\mathbf{p}, u)$'s are Hicksian demand function. A property of any system of Hicksian demand functions is that

$$\sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} = 0 \qquad (1-6)$$

for any good i. Equation $(1-6)$ can be written in matrix form

$$\mathbf{p}'H = 0 \qquad (1-7)$$

which implies that the matrix H does not have full rank. That is, the vector $\mathbf{p}' \neq \mathbf{0}'$ is an **eigenvector** to the matrix H, corresponding to an eigenvalue of 0. (A vector $\mathbf{v}'$ is an eigenvector to a matrix H, with an eigenvalue λ, if and only if $\mathbf{v}'H = \lambda\mathbf{v}'$.) Any matrix with an eigenvalue of 0 must have a determinant of 0.

The property $(1-6)$ must hold for any system of Hicksian demand functions. But it is not an independent property which must be checked in addition to homogeneity of degree 0, symmetry and negative semi–definiteness. It follows from homogeneity of degree 0. If the function $f(t)$ is defined by

$$f(t) = x_i(t\mathbf{p}, u) \qquad (1-8)$$

then

$$f'(t) = \sum_{j=1}^{n} p_j \frac{\partial x_j}{\partial p_i} \qquad (1-9)$$

Since homogeneity of degree 0 implies that $f(t)$ does not vary with t, homogeneity of degree 0 means that equation $(1-9)$ implies property $(1-6)$.]

*Q*2. The following table lists the prices of 3 goods, and the quantities a consumer chose of the goods, in 4 different years.

From these data, what can be concluded about how well off the consumer was in the different years? Explain briefly.

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	1	5	2	3
2	3	1	2	3	12	4
3	1	2	1	5	1	6
4	3	2	5	2	6	1

*A*2. Using these data, the costs of the bundles in the different years can be calculated. If C_i^t denotes the cost of bundle x_i in year t, then the costs are

year	C_1^t	C_2^t	C_3^t	C_4^t
1	10	19	12	9
2	23	29	28	14
3	12	31	13	15
4	34	53	47	23

(For example, the number 28 in the third column of the second row indicates that bundle $\mathbf{x}^3$ costs 28 when evaluated using year–2 prices $\mathbf{p}^2$.)

The first row reveals that bundle $\mathbf{x}^1$ is preferred to bundle 4 : the cost, \$9, of bundle is 4 is less than the cost, \$10, of the bundle $\mathbf{x}^1$ which she actually chose in year 1. The first row reveals nothing about how she values bundles $\mathbf{x}^2$ and $\mathbf{x}^3$, compared to $\mathbf{x}^1$, since these 2 bundles cost more than \$10 : since she could not afford $\mathbf{x}^2$ or $\mathbf{x}^3$ in year 1, the fact that she actually chose $\mathbf{x}^1$ does not indicate whether she actually preferred $\mathbf{x}^1$ to either of these bundles.

In year 2, she chose $\mathbf{x}^2$, but could have afforded any of the other 3 bundles (which all cost less than the cost, \$29, of $\mathbf{x}^2$). So her behaviour in year 2 reveals that she prefers the bundle $\mathbf{x}^2$ to any of the other 3 bundles.

In year 3, she could have afforded $\mathbf{x}^1$, so her behaviour reveals that she prefers the bundle she did choose, $\mathbf{x}^3$, to $\mathbf{x}^1$.

Finally, in year 4, the bundle she actually chose, $\mathbf{x}^4$, is cheaper than any of the other three bundles, so that her behaviour in year 4 reveals nothing about her preferences among the 4 bundles.

So her behaviour is consistent with the strong axiom of revealed preference, and indicates that her preference ordering must be $\mathbf{x}^2 \succ \mathbf{x}^3 \succ \mathbf{x}^1 \succ \mathbf{x}^4$.

*Q*3. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at three different times, and the quantities x^t of the 3 goods chosen at the three different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (4, 4, 6)$ when the price vector was $\mathbf{p} = (1, 3, 5)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	5	3	1	8	2	2
2	1	3	5	4	4	6
3	2	4	2	3	5	3
4	2	5	5	6	2	5

*A*3. As in the previous question, the costs of the bundles in the different years can be calculated. If C_i^t denotes the cost of bundle x_i in year t, then the costs are

year	C_1^t	C_2^t	C_3^t	C_4^t
1	48	38	33	41
2	24	46	33	37
3	28	36	32	30
4	36	58	46	47

The first row of the table of costs shows that the bundle $\mathbf{x}^1$ is directly revealed preferred to each of the other three bundles : $\mathbf{x}^1$ costs \$48 in the year in which it was chosen, more than the cost of any other bundle. So the person could have afforded any of the bundles in year 1, and the fact that she chose $\mathbf{x}^1$ reveals (directly) that the bundle is preferred to any other.

But the second row shows that $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^1$, $\mathbf{x}^3$ and $\mathbf{x}^4$: $\mathbf{x}^2$ costs more, at year 2 prices, than any other bundle.

So the first two rows of the table imply a violation of WARP : row 1 has $\mathbf{x}^1$ directly revealed preferred to $\mathbf{x}^2$ and row 2 has $\mathbf{x}^2$ directly revealed preferred to $\mathbf{x}^1$.

The third row indicates that $\mathbf{x}^3$ is directly revealed preferred to $\mathbf{x}^1$ and to $\mathbf{x}^4$ (but not to $\mathbf{x}^2$), since it costs more than those two bundles. So another violation of WARP : row 1 has $\mathbf{x}^1$ directly revealed preferred to $\mathbf{x}^3$ and row 3 has $\mathbf{x}^3$ directly revealed preferred to $\mathbf{x}^1$.

The fourth row indicates that $\mathbf{x}^4$ is directly revealed preferred to $\mathbf{x}^1$ and to $\mathbf{x}^3$ (but not to $\mathbf{x}^2$), since it costs more than those two bundles. So two more violations of WARP : row 1 has $\mathbf{x}^1$ directly revealed preferred to $\mathbf{x}^4$ and row 4 has $\mathbf{x}^4$ directly revealed preferred to $\mathbf{x}^1$; row 3 has $\mathbf{x}^3$ directly revealed preferred to $\mathbf{x}^4$ and row 4 has $\mathbf{x}^4$ directly revealed preferred to $\mathbf{x}^3$.

There are 6 possible comparisons of bundles : ($\mathbf{x}^1$ versus $\mathbf{x}^2$, $\mathbf{x}^1$ versus $\mathbf{x}^3$, $\mathbf{x}^1$ versus $\mathbf{x}^4$, $\mathbf{x}^2$ versus $\mathbf{x}^3$, $\mathbf{x}^2$ versus $\mathbf{x}^4$, and $\mathbf{x}^3$ versus $\mathbf{x}^4$). 4 of these comparisons violate WARP here. The remaining 2 comparisons do not violate WARP, but they do violate SARP : $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^3$, but $\mathbf{x}^3$ is directly revealed preferred to $\mathbf{x}^1$ which is directly revealed preferred to $\mathbf{x}^2$ (and so $\mathbf{x}^3$ is indirectly revealed preferred to $\mathbf{x}^2$) ; $\mathbf{x}^2$ is directly revealed preferred to $\mathbf{x}^4$, but $\mathbf{x}^4$ is directly revealed preferred to $\mathbf{x}^1$ which is directly revealed preferred to $\mathbf{x}^2$ (and so $\mathbf{x}^4$ is indirectly revealed preferred to $\mathbf{x}^2$).

So every possible comparison of consumption bundles here violates SARP.

$Q4$. Thelma and Louise are both risk–averse expected utility maximizers. Thelma has a utility–of–wealth function

$$U(W) = \sqrt{W} - \frac{8}{W}$$

while Louise has a utility–of–wealth function

$$V(W) = 2 - \frac{4}{\sqrt{W}}$$

Both Thelma and Louise have an initial wealth of $W = 4$.

Find a simple (2–state) risky proposition which Thelma would accept but which Louise would reject. Find a simple (2–state) risky proposition which Louise would accept but which Thelma would reject.

$A4$. Both Thelma and Louise have the same level of utility, 0, at their initial wealth level of 4. So each would accept a gamble if and only if it leads to a positive level of expected utility.

If the two people's coefficients of relative risk aversion are computed, for Thelma

$$R_r^T = \frac{W^2 + 64\sqrt{W}}{2W^2 + 32\sqrt{W}} \tag{4-1}$$

$$R_r^L = \frac{3}{2} \tag{4-2}$$

so that we cannot say that one person is unequivocally more risk–averse than the other. Evaluated at the initial wealth of 4, both Thelma and Louise have the same coefficient of relative risk aversion. But this is only true exactly at $W = 4$: if $W < 4$ then $R_r^T > R_r^L$ but if $W > 4$ then $R_r^L > R_r^T$.

The fact that Thelma's coefficient of relative risk aversion is greater than Louise's when wealth is small, but bigger when wealth is large suggests that Thelma values more undertakings with a small probability of a large payoff, and Louise would value more undertakings with a larger probability of a modest payoff. Alternatively, note that Louise's utility of wealth cannot exceed 2, no matter how high her wealth, whereas Thelma's utility gets very large as her wealth grows large.

For example, consider the gamble $g^1 = (0.8 \circ 9, 0.2 \circ 1))$. Thelma's expected utility from this gamble is

$$EU(g^1) = -(0.2)(14) + (0.8)(\frac{19}{9} < 0$$

so that she would reject the gamble ; it offers her a lower level of expected utility than 0, the level at her initial wealth. Louise's expected utility is

$$EV(g^1) = -(0.2)(2) - (0.8)(\frac{2}{3} > 0$$

so that Louise gets a higher level of expected utility from g^1 than from her initial certain wealth 4. Thus Louise would accept the gamble g^1 and Thelma would reject it.

But now consider the gamble $g^2 = (0.5 \circ 400, 0.5 \circ 1)$. For Thelma

$$EU(g^2) = -(0.5)(14) + (0.5)(19.98) > 0$$

so that she would accept the gamble. For Louise

$$EV(g^2) = -(0.5)(2)) + (0.5)(1.8) < 0$$

so that she would reject the gamble.

Notice that both gambles increase the people's expected wealth (from 4 to 7.4 for gamble 1, and from 4 to 200.5 for gamble 2) ; otherwise neither person would accept either gamble. Gamble 1 offers a large probability of a modest gain, but a small probability of a large loss : Thelma rejects the gamble because her utility of wealth declines rapidly as her wealth gets very small. Gamble 2 offers a smaller chance of winning than gamble 1, but a much better prize. Since Louise's utility of wealth increases very very slowly as her wealth increases to high levels, she gets less expected utility from this gamble.

*Q*5. Suppose that an expected utility maximizer has a utility–of–wealth function

$$U(W) = \frac{1}{1-\beta} W^{1-\beta} \quad \beta \geq 0$$

The person has an initial wealth of W_0. She has the opportunity to invest (exactly) half of her initial wealth in a speculative stock : with probability π the stock will triple in value, but with probability $1-\pi$ the stock will be worthless.

Given the person's preference parameter β, and given her initial wealth, what must π be in order to induce her to invest half of her wealth in the stock?

*A*5. If she does not invest, she has a certain wealth level of W_0, so that her expected utility of wealth is

$$U(W_0) = \frac{1}{1-\beta} W_0^{1-\beta} \qquad (5-1)$$

If she does invest, then with probability $1-\pi$ the stock will be worthless. But she will still have wealth of $(0.5)W_0$, since she invested only half her wealth in the stock. But with probability π she will have wealth of $2W_0$, since her initial investment will triple in value, giving her $(1.5)W_0$ to add to the initial half of her wealth which she did not invest.

So notice that – if she were to choose to invest — there are only two possible outcomes : if the stock fails (which happens with probability $1-\pi$), her wealth will be $W_0/2$; if the stock does well (which happens with probability π), her wealth will be $W_0/2 + 3(W_0/2) = 2W_0$. That is, investing half her wealth in the stock can be represented by the gamble

$$g \equiv (\pi \circ 2W_0, (1-\pi) \circ \frac{W_0}{2})$$

So her expected utility from investing in the stock is

$$EU = (1-\pi)\frac{1}{1-\beta}[(0.5)W_0]^{1-\beta} + \pi \frac{1}{1-\beta}[2W_0]^{1-\beta} \qquad (5-2)$$

To make her undertake the investment, the probability of success π must be high enough so that expression $(5-2)$ is at least as large as expression $(5-1)$.

If

$$\frac{1}{1-\beta}W_0^{1-\beta} = (1-\pi)\frac{1}{1-\beta}[(0.5)W_0]^{1-\beta} + \pi\frac{1}{1-\beta}[2W_0]^{1-\beta} \qquad (5-3)$$

then

$$(1-\pi)(0.5)^{1-\beta} + \pi(2)^{1-\beta} = 1 \qquad (5-4)$$

(dividing both sides of $(5-3)$ by $(1-\beta)W_0^{1-\beta}$).

So the value of π which makes the person just willing to undertake the investment is the value of π which satisfies equation $(5-4)$. Notice that this value does not depend on her initial wealth W_0 : since the person has a constant coefficient of relative risk aversion, and since the investment involves a constant fraction (half) of her wealth, her choice of whether or not to undertake the investment depends only on her coefficient of relative risk aversion β and on the probability π of success.

Equation $(5-4)$ can be written

$$\pi = \frac{1-(0.5)^{1-\beta}}{2^{1-\beta}-(0.5)^{1-\beta}} \qquad (5-5)$$

or

$$\pi = \frac{2^{1-\beta}-1}{[2^{1-\beta}]^2-1} \qquad (5-6)$$

(where I have used the fact that $2^a = (0.5)^{-a}$).

Since $b^2-1=(b+1)(b-1)$, equation $(5-6)$ can be written

$$\pi = \frac{1}{2^{1-\beta}+1} \qquad (5-7)$$

(letting $b = 2^{1-\beta}$).

The solution to equation $(5-7)$ is an increasing function of β. It has to be : the higher β is, the more risk–averse the person is, and the greater the probability of success has to be to induce her to invest. As β increases from 0 to ∞, the required probability of success π increases from 1/3 to 1.

[The original expression $U(W) = \frac{1}{1-\beta}W^{1-\beta}$ defines an increasing concave function of W for any $\beta > 0$. Except it is not defined at $\beta = 1$. In this limiting case, $U(W) = \ln W$, and equation $(5-7)$ still defines the threshold probability π which will induce her to undertake the stock investment.]