$Q1$. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of 60 units of good 1 and 29 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = \ln x_1^1 + \ln x_2^1$$

$$u^2(x_1^2, x_2^2) = \ln x_1^2 + x_2^2$$

where x_j^i is person i's consumption of good j?

$A1$. An allocation (in the interior of the Edgeworth box) which is Pareto optimal must have the two people's marginal rates of substitution equal. Given the preferences of the 2 people

$$MRS^1 \equiv \frac{u_1^1}{u_2^1} = \frac{x_2^1}{x_1^1} \tag{1-1}$$

$$MRS^2 \equiv \frac{u_1^2}{u_2^2} = \frac{1}{x_1^2} \tag{1-2}$$

Since $x_1^2 = 60 - x_1^1$, the condition that $MRS^1 = MRS^2$ can be written

$$\frac{x_2^1}{x_1^1} = \frac{1}{60 - x_1^1} \tag{1-3}$$

so that

$$x_2^1 = \frac{x_1^1}{60 - x_1^1} \tag{1-4}$$

Equation $(1-4)$ is the equation defining the contract curve – at least for allocations for which $\mathbf{x}^1 >> 0$ and $\mathbf{x}^2 >> 0$. It goes through the bottom–left corner of the Edgeworth box : $x_2^1 = 0$ when $x_1^1 = 0$. It slopes up :

$$\frac{\partial x_2^1}{\partial x_1^1} = \frac{60}{(60 - x_1^1)^2} \tag{1-5}$$

It starts out below the diagonal of the Edgeworth box, since the contract curve defined by $(1-4)$ has a slope of $1/60$ at $x_1^1 = x_2^1 = 0$. But then it gets steeper, and it crosses the diagonal at the point at which $x_2^1/x_1^1 = 29/60$, or

$$\frac{1}{60 - x_1^1} = \frac{29}{60} \tag{1-7}$$

which happens when $x_1^1 \approx 59.97$.

The contract curve actually hits the top of the Edgeworth box, since equation $(1-4)$ implies that $x_2^1 \to \infty$ as $x_1^1 \to 60$, which cannot happen if the curve is to stay inside the box. Equation $(1-4)$ implies that $x_2^1 = 29$ when

$$29 = \frac{x_1^1}{60 - x_1^1} \tag{1-8}$$

or

$$x_1^1 = 58$$

So the contract curve is the solution to equation $(1-4)$ for $0 \leq x_1^1 \leq 58$, and then the set of all allocations $(x_1^1, 60)$ with $58 \leq x_1^1 \leq 60$.

*Q*2. What are all the allocations in the core of a 3–person, 2–good economy, in which each person's preferences can be represented by the utility function

$$u^i(x_1^i, x_2^i) = 100 - \frac{1}{x_1^i} - \frac{1}{x_2^i}$$

where x_j^i is person i's consumption of good j, and where the endowments e^i of the three people are $e^1 = (2,0)$, $e^2 = (0,2)$, $e^3 = (0,2)$?

*A*2. A necessary condition for an allocation to be in the core is that the allocation be Pareto optimal. The Pareto optimal allocations are those for which $MRS^1 = MRS^2 = MRS^3$. Given that

$$MRS^i \equiv \frac{u_1^i}{u_2^i} = [\frac{x_2^i}{x_1^i}]^2 \qquad (2-1)$$

Pareto optimality requires that

$$[\frac{x_2^1}{x_1^1}]^2 = [\frac{x_2^2}{x_1^2}]^2 = [\frac{x_2^3}{x_1^3}]^2 \qquad (2-2)$$

which implies that the ratio of good–1 consumption to good–2 consumption must be the same for all 3 people, if the allocation is Pareto optimal. Since the aggregate endowments of the three people are 2 units of good 1 and 4 units of good 2, Pareto optimality requires an allocation such that

$$x_2^i = 2x_1^i \quad i = 1, 2, 3 \qquad (2-3)$$

So any allocation in the core must be of the form $\mathbf{x}^1 = (a, 2a)$, $\mathbf{x}^2 = (b, 2b)$, $\mathbf{x}^3 = (c, 2c)$, with $a + b + c = 2$.

Second, any allocation in the core must be just as good for each person as her own endowment. But here, this "individual rationality" does not restrict the allocations. Since each person's utility approaches $-\infty$ as $x_1^i \to 0$ or $x_2^i \to 0$, any allocation $\{(a, 2a), (b, 2b), (c, 2c)\}$ will be preferred by all 3 people to their initial endowments, as long as a, b and c are all positive.

Third, a cannot be too small. Suppose first that person 1 gets $(a, 2a)$, and person 2 and person 3 each get the same allocation : $(b, 2b) = (\frac{2-a}{2}, \frac{4-2a}{2})$. Person 1 could consider forming a coalition with person 2. This new coalition would have a total endowment of $(2, 2)$, and so should give both people the same consumption of each good if it divides this endowment efficiently. So person 1 would get (x, x). Person 1 would be made just as well off by this new coalition if

$$100 - \frac{1}{x} - \frac{1}{x} = 100 - \frac{1}{a} - \frac{1}{2a} \qquad (2-3)$$

or

$$x = \frac{4}{3}a \tag{2-4}$$

What does that do for the other person in the coalition, person 2? He gets what is left, $(2-x, 2-x)$. This is better for him if

$$100 - \frac{1}{2-x} - \frac{1}{2-x} \geq 100 - \frac{1}{b} - \frac{1}{2b} = 100 - \frac{2}{2-a} - \frac{2}{4-2a} \tag{2-5}$$

which is equivalent to

$$\frac{6}{4-2a} > \frac{2}{2-x}$$

or

$$2a > 3x - 2$$

Substituting from $(2-4)$ for x, this new coalition can make person 1 just as well off, and person 2 better off, if and only if

$$2a > 4a - 2$$

or

$$a < 1 \tag{2-6}$$

So an allocation $\{(a, 2a), (b, 2b), (b, 2b)\}$ will be in the core if and only if $a \geq 1$; if person 1 gets a worse consumption vector than $(1, 2)$, she can form a coalition with one of the other two people, and make both of them better off.

Finally, what would happen if person 2 and person 3 were treated differently? Suppose the allocation is $\{(a, 2a), (b, 2b), (c, 2c)\}$. If $b < c$, a coalition of person 1 and person 2 might try and block the allocation. This coalition of person 1 and person 2, dividing its endowment of $(2, 2)$ efficiently, would give (x, x) to person 1 and $(2-x, 2-x)$ to person 2. To give person 1 the same utility as she got in the original allocation, again condition $(2-4)$ must hold. But then person 2 will gain from joining the coalition if and only if

$$100 - \frac{1}{2-x} - \frac{1}{2-x} > 100 - \frac{1}{b} - \frac{1}{2b} \tag{2-7}$$

Condition $(2-7)$ is equivalent to

$$2 - x > \frac{4b}{3} \tag{2-8}$$

When x satisfies $(2-4)$ condition $(2-8)$ becomes

$$a < \frac{3}{2} - b \tag{2-9}$$

If $b < c$, then the allocation $\{(a, 2a), (b, 2b), (c, 2c)\}$ can be blocked by a coalition of people 1 and 2 whenever $(2-8)$ holds ; similarly, if $c < b$, then the allocation $\{(a, 2a), (b, 2b), (c, 2c)\}$ can be blocked by a coalition of people 1 and 3 whenever

$$a < \frac{3}{2} - c \tag{2-10}$$

So what allocations are in the core? The allocations have to be of the form $\{(a, 2a), (b, 2b), (c, 2c)\}$, with $a + b + c = 2$ to be Pareto optimal. And to avoid being blocked, it must be true that

$$a \geq \frac{3}{2} - \min(b, c) \qquad (2-11)$$

If $c \geq b$, equation $(2-11)$ becomes $a \geq \frac{3}{2} - b$, or

$$a + b \geq \frac{3}{2} \qquad (2-12)$$

which is the same thing as

$$c \leq \frac{1}{2} \quad \text{if} \quad c \geq b \qquad (2-13)$$

Similarly, if $b \geq c$, then $(2-11)$ becomes

$$b \leq \frac{1}{2} \quad \text{if} \quad b \geq c \qquad (2-14)$$

So combining $(2-13)$ and $(2-14)$, if the allocation is in the core than it must be the case that

$$\max(b, c) \leq \frac{1}{2} \qquad (2-15)$$

The allocations in the core are all the allocations $(a, 2a), (b, 2b), (c, 2c)$ with $a + b + c = 2$, provided that $b \leq \frac{1}{2}$ and $c \leq \frac{1}{2}$.

*Q*3. What would the competitive equilibrium be in the economy described in question #1 above, if person 1's endowment of goods 1 and 2 was $\mathbf{e}^1 = (40, 18)$ and person 2's endowment was $\mathbf{e}^2 = (20, 11)$?

*A*3. The prices (p_1, p_2) are equilibrium prices if they make the sum of the two people's excess demands for each good equal 0.

To find the excess demands, we need the two people's demand functions.

It is probably easiest to look at the demand functions for good 1.

Person 1 has Cobb–Douglas preferences, so that her total demand for good 1 is

$$x_1^1(p_1, p_2, y^1) = \frac{y^1}{2p_1} \qquad (3-1)$$

where y^1 is her income. Since her endowment is $(40, 18)$, therefore

$$y^1 = 40p_1 + 18p_2$$

implying that her demand function for good 1 is

$$x_1^1(p_1, p_2) = \frac{40p_1 + 18p_2}{2p_1} = 20 + 9\frac{p_2}{p_1} \qquad (3-2)$$

Since her excess demand for good 1 is her total demand for the good, minus her endowment, therefore

$$z_1^1(p_1, p_2) = 20 + 9\frac{p_2}{p_1} - 40 = 9\frac{p_2}{p_1} - 20 \qquad (3-3)$$

Person 2's first–order condition for utility maximization is

$$MRS^2 = \frac{1}{x_1^2} = \frac{p_1}{p_2} \qquad (3-4)$$

Equation $(3-4)$ can be re–arranged to define his total demand for good 1,

$$x_1^2 = \frac{p_2}{p_1} \qquad (3-5)$$

Since person 2 has quasi–linear preferences, his demand for good 1 does not depend on his income. From $(3-5)$, and the fact that his endowment of good 1 is 20, his excess demand function for good 1 is

$$z_1^2(p_1, p_2) = \frac{p_2}{p_1} - 20 \qquad (3-6)$$

The market for good 1 will clear if total excess demands are zero. From $(3-3)$ and $(3-6)$

$$Z_1(p_1, p_2) \equiv z_1^1(p_1, p_2) + z_1^2(p_1, p_2) = 10\frac{p_2}{p_1} - 40 \qquad (3-7)$$

Equation $(3-7)$ says that the total excess demand for good 1 will be zero if and only if

$$\frac{p_2}{p_1} = 4 \qquad (3-8)$$

So any price vector of the form $(p, 4p)$ will be an equilibrium price vector for this economy.

From equation $(3-2)$, person 1's total demand for good 1 will be $20+9(4) = 56$ in equilibrium, and from equation $(3-5)$ person 2's total demand for good 1 is 4. Person 2's total income is $20p + 4(11)p$ in equilibrium ; she spends $4p$ on good 1, leaving her with $20p + 44p - 4p = 60p$ to spend on good 2. Since the price of good 2 is $4p$, that means that he buys 15 units of good 2. Person 1's income is $40p + 18(4p) = 112p$ at equilibrium prices. Because of her Cobb–Douglas preferences, she spends half her income on good 2, so she spends $56p$ on good 2. At a price for good 2 of $4p$, that means person 1 consumes 14 units of good 2 at equilibrium prices.

So the competitive equilibrium allocation is

$$\mathbf{x}^1 = (56, 14) \quad ; \quad \mathbf{x}^2 = (4, 15)$$

*Q*4. What is the competitive equilibrium to the economy described in question #2?

*A*4. Each person in this economy has a marginal rate of substitution defined by

$$MRS^i = \frac{u_1^i}{u_2^i} = [\frac{x_2^i}{x_1^i}]^2 \qquad (4-1)$$

In any competitive equilibrium, each person i will choose a consumption bundle for which her MRS^i equals the price ratio p_1/p_2.

So equation $(4-1)$ implies that each person must consume goods 1 and 2 in the same ratio : for each person

$$\frac{x_2^i}{x_1^i} = \sqrt{\frac{p_1}{p_2}} \qquad (4-2)$$

Since the total quantity of good 1 available is 2, and the total quantity of good 2 available is 4, that means that each person must consume twice as much good #2 as good #1 in any competitive equilibrium :

$$\frac{x_2^i}{x_1^i} = 2 \quad i = 1, 2, 3 \qquad (4-3)$$

in any competitive equilibrium.

(Of course equation $(2-1)$ said that $x_2^i = 2x_1^i$ in any Pareto optimum, so that same condition should hold in any competitive equilibrium.)

Equations $(4-3)$ and $(4-2)$ therefore imply that the equilibrium price ratio must be such that $\sqrt{p_1/p_2} = 2$, so that the equilibrium prices for this economy must be of the form $(p_1, p_2) = (4p, p)$ for some positive p.

Since person 1 has an endowment vector $(2, 0)$, her income is $8p$ in equilibrium. She consumes a consumption bundle $(a, 2a)$, which must have a cost equal to her income. So she consumes the bundle $(a, 2a)$, where

$$(4p)a + p(2a) = 8p \qquad (4-3)$$

implying that

$$a = \frac{4}{3} \qquad (4-4)$$

Person 2 has an income of $2p$, since his endowment is $(0, 2)$, so that his equilibrium consumption bundle $(b, 2b)$ satisfies the equation

$$(4p)b + p(2b) = 2p \qquad (4-5)$$

or

$$b = \frac{1}{3} \qquad (4-6)$$

Since person 3 has the same endowment as person 2, her consumption bundle is also $(\frac{1}{3}, \frac{2}{3})$.

Therefore, the equilibrium allocation is

$$\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\} = \{(\frac{4}{3}, \frac{8}{3}), (\frac{1}{3}, \frac{2}{3}), (\frac{1}{3}, \frac{2}{3})\} \qquad (4-7)$$

As it must be, this equilibrium is in the core of this economy.

*Q*5. Find all the Nash equilibria in the following strategic–form two–person game.

	a	b	c	d	e	f
A	$(3,0)$	$(12,5)$	$(7,2)$	$(4,19)$	$(2,16)$	$(3,14)$
B	$(2,0)$	$(1,20)$	$(8,0)$	$(2,2)$	$(4,8)$	$(1,18)$
C	$(6,12)$	$(0,4)$	$(4,5)$	$(6,6)$	$(8,6)$	$(5,7)$
D	$(4,18)$	$(3,5)$	$(0,8)$	$(8,6)$	$(5,12)$	$(4,20)$

*A*5. This game can be solved by repeated elimination of strictly dominated strategies.

First, note that column c is strictly dominated for player 2 by column e : no matter what player 1 does, player 2 gets a higher payoff from playing e than he does from playing c.

So we can cross out column c : player 2 will never choose to play it if he is rational.

Once column c has been eliminated, player 1 has a strictly dominated strategy in this new, smaller, 4–by–5 game. Now row B is dominated strictly by row D. Again, whatever action player 2 chooses, player 1 gets a higher payoff from choosing D than from choosing B. [This is not true if player 2 were to play column c, but we have already eliminated that column, since player 2 would never want to play it.]

Now player 2 has another strictly dominated strategy in the smaller 3–by–5 game. Column b is now dominated strictly by column f.

With columns b and c gone, row A is now dominated strictly by row C for player 1 ; row A can be eliminated.

With rows A and B eliminated, column e is now dominated strictly by column f for player 2 ; column e can be eliminated.

Column e can also now be eliminated ; it is dominated strictly by column f as well.

With only columns a and f left for player 2, row C strictly dominates row D for player 1.

And if player 1 chooses row C, player 2's best response is to choose column a.

So (C, a) is a pure–strategy Nash equilibrium to this game.

But it is also the **only** Nash equilibrium (in pure or mixed strategies) to this game : no player will ever choose to play a strategy with positive probability if it can be eliminated by strict dominance.