GS/ECON 5010 Answers to Assignment 4 December 2010

Q1. Suppose that the aggregate demand curve by men for some good has the equation

$$Q^M = 150 - p$$

where Q^M is the aggregate quantity demanded by men, and p the price they pay. The aggregate demand curve of women, for the same product, is

$$Q^W = 300 - 5p$$

where Q^W is the aggregate quantity demanded by women.

A monopoly is able to supply the good at zero cost (in unlimited quantities). Compare the price paid by men, the price paid by women, and the monopoly's profit in the following two situations :

(i) The monopoly can charge different prices to men and women (who are not able to resell the good).

(ii) The monopoly must charge the same price to all buyers.

A1. The behaviour in part (i) is sometimes called "third–degree price discrimination", and is discussed in a little more detail in Varian's (undergraduate and graduate) textbooks, for example.

(i) A monopoly should set $MR = MC$ in each market served, if it wants to maximize profits. Here $MC = 0$. To calculate the MR in the men's market, convert the demand function $Q^M = 150 - p$ into an inverse demand function, expressing the price charged to men as a function of the quantity sold to men

$$p^M = 150 - Q^M \quad (1-1)$$

If the demand curve is linear, then the marginal revenue curve has exactly twice the slope of the demand curve, and starts out with the same intercept, so that

$$MR^M = 150 - 2Q^M \quad (1-2)$$

Equation $(1-2)$ may be obtained directly by calculating the derivative of revenue – $R(Q^M) = p(Q^M)Q^M$ — with respect to Q^M after substituting from $(1-1)$ for $p(Q^M)$.

From $(1-2)$, the monopoly's optimal quantity to sell to men, for which $MR^M = MC = 0$, is $Q^M = 75$, resulting (from $(1-1)$ in a price for men of $p^M = 75$.

Similarly, the inverse demand curve for women is

$$p^W = 60 - Q^W/5 \quad (1-3)$$

with associated marginal revenue function

$$MR^W = 60 - 2Q^W/5 \quad (1-4)$$

so that setting $MR^W = 0$ results in a quantity $Q^W = 150$ being sold to women, at a price of $p^W = 30$.

[The answers can also be obtained directly, by having the monopoly choose prices p^M and p^W so as to maximize $p^M(150 - p^M) + p^W(300 - 5p^W)$.]

The monopoly makes total profits of $p^M Q^M + p^W Q^W = 5625 + 4500 = 10125$.

(ii) The aggregate demand, if the monopoly charges the same price to everyone, is

$$Q = Q^M + Q^W = (150 - p) + (300 - 5p) = 450 - 6p \qquad (1-5)$$

(Note that equation $(1-5)$ applies only when price is so low that some people of either sex choose to buy : if $p > 60$, then no women will choose to buy.)

The inverse demand curve corresponding to $(1-5)$ is

$$p = 75 - Q/6 \qquad (1-6)$$

so that

$$MR = 75 - Q/3 \qquad (1-7)$$

and setting $MR = MC = 0$ implies (from equation $(1-7)$) a quantity of $Q = 225$ being sold, at a price of $p = 37.5$.

[Again, this can be obtained directly from the maximization of $pQ(p)$, where $(1-5)$ defines $Q(p)$.]

The monopoly now makes a profit of $(225)(37.5) = 8437.50$.

Here, allowing third–degree price discrimination increases the monopoly's profits. This must always be the case : adding the constraint $p^W = p^M$ into the monopoly's profit maximization cannot increase the value of its maximized profit, and usually will decrease it.

Here third–degree price discrimination makes one group of buyers (here, women) better off, by lowering the price they pay, and makes the other group of buyers worse off. The characteristic of men here which makes them victims of price discrimination is their lower own–price–elasticity of demand.

In this example, price discrimination actually does not change the total quantity sold : in part (i), $Q^M + Q^W = 225$. This result holds whenever demand curves are linear, and marginal cost is constant.

Although 2 groups benefit here from price discrimination (women, and the monopoly's owners), and only one group loses (men), it turns out here that the loss of consumer surplus under price discrimination, added up over all men, actually exceeds the sum of the increase in women's consumer surplus, and the increased profits of the monopoly. Again this is true whenever demands are linear and marginal costs constant : price discrimination shifts sales from people who value the good most (men here) to those who value it least.

Q2. What is the equilibrium if Cournot duopolists, producing a homogeneous good, face an inverse demand curve

$$p = 10 - Q$$

(where Q is the aggregate quantity produced by the two firms, and p the resulting market price), if each firm's total cost of production is

$$TC(q) = q^2 + F \qquad \text{if} \qquad q > 0$$

(and $TC(0) = 0$), where $F > 0$ is each firm's positive fixed cost?

A2. Except for the fixed costs, this is just a special case of a problem from an earlier assignment : problem 3 of assignment 3 from winter 2005.

If a firm has chosen to produce at a positive level, then the fixed costs should not affect its choice of output. Firm 1 chooses an output q_1 so as to maximize

$$(10 - q_1 - q_2)q_1 - (q_1)^2 - F \tag{2 – 1}$$

taking the other firm's output q_2 as given. From the first–order condition for the maximization of this profit

$$10 - q_2 - 4q_1 = 0 \tag{2 – 2}$$

the firm's reaction function is

$$q_1 = \frac{10 - q_2}{4} \tag{2 – 3}$$

Similarly, firm 2 has a reaction function

$$q_2 = \frac{10 - q_1}{4} \tag{2 – 4}$$

if it chooses a positive output level. The only possible Cournot equilibrium which satisfies both $(2-3)$ and $(2-4)$ is

$$q_1 = q_2 = 2 \tag{2 – 5}$$

If each firm produces 2 units, then the market output is $Q = 4$, and the market price is $10 - 4 = 6$. Each firm earns a profit of

$$\pi = pq - q^2 - F = (6)(2) - 2^2 - F = 8 - F \tag{2 – 6}$$

So if fixed costs are low enough —$F < 8$ — the Cournot equilibrium has $q_1 = q_2 = 2$, and $p = 6$.

But if $F > 8$, a firm would do better producing nothing (and avoiding the fixed costs), than producing 2 and making negative profits. So $q_1 = q_2 = 2$ is a Cournot equilibrium if and only if $F \leq 8$.

If $F > 8$, at least one firm will want to shut down. If $q_2 = 0$, then firm 1 will want to choose q_1 so as to maximize $(10 - q_1)q_1 - (q_1)^2 - F$. The solution to this maximization is for firm 1 to choose

$$q_1 = 2.5$$

If $q_1 = 2.5$ and $q_2 = 0$, then $p = 7.5$ and firm 1's net profits are

$$\pi_1 = (2.5)(7.5) - (2.5)^2 - F = 12.5 - F \qquad (2-7)$$

But will firm 2 be willing to choose an output of 0 if $q_1 = 2.5$?

From equation $(2-4)$, firm 2's best reaction to $q_1 = 2.5$ — conditional on its producing a positive level of output — is

$$q_2 = \frac{10 - 2.5}{4} = 1.875 \qquad (2-8)$$

Then industry output will be $2.5 + 1.875 = 4.375$, and the market price will be 5.625. Firm 2's profit will be

$$\pi_2 = (1.875)(5.625) - (1.875)^2 - F = (1.875)(3.75) - F = 7.03125 - F \qquad (2-8)$$

Equation $(2-8)$ shows that, as long as $F > 7.03125$, firm 2 would rather produce nothing at all, should firm 1 choose to produce $q_1 = 2.5$.

So, if fixed costs are high, there are asymmetric Cournot equilibria, in which one firm produces an output level of 2.5, which is a single–price monopolist's preferred production level, and in which the other firm chooses to produce nothing at all. As long as $12.5 \geq F \geq 7.03125$, the firm producing $q_i = 2.5$ will make positive profits, and the other firm cannot react profitably.

That means that there are several Cournot equilibria (in pure strategies) for some levels of fixed costs.

(i) if $F < 7.03125$, the unique Cournot equilibrium is $q_1 = q_2 = 2$

(ii) if $7.03125 < F < 8$, there are several Cournot equilibria (in pure strategies) : $q_1 = q_2 = 2$, $q_1 = 2.5, q_2 = 0$, and $q_1 = 0, q_2 = 2.5$

(iii) if $8 < F \leq 12.5$, the Cournot equilibria are asymmetric : $q_1 = 2.5, q_2 = 0$ and $q_1 = 0, q_2 = 2.5$

(iv) if $F > 12.5$, the only Cournot equilibrium is for both firms to shut down

note : the fixed costs are a pretty important part of this question ; noticing that $q_1 = q_2 = 2$ is a Nash equilibrium only if $F \leq 8$ was worth 5 marks

Q3. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of 5 units of good 1 and 20 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = x_1^1 + 10x_2^1 - \frac{1}{2}[x_2^1]^2$$

$$u^2(x_1^2, x_2^2) = 10x_1^2 + x_2^2 - \frac{1}{2}[x_1^2]^2$$

where x_j^i is person i's consumption of good j?

A3. Here good 2 becomes a bad for person 1, if $x_2^1 > 10$, and good 1 becomes a bad for person 2, if $x_1^2 > 10$. But this will never happen in a Pareto optimal allocation : if $x_2^1 > 10$ ($x_1^2 > 10$) then the utility of both people good be increased by transferring a little of good 2 from person 1 to person 2 (transferring a little of good 1 from person 2 to person 1).

Person 1's MRS is

$$MRS^1 = \frac{u_1^1}{u_2^1} = \frac{1}{10 - x_2^1} \qquad (3-1)$$

and person 2's MRS is

$$MRS^2 = \frac{u_1^2}{u_2^2} = \frac{10 - x_1^2}{1} \qquad (3-2)$$

Since efficiency requires that $MRS^1 = MRS^2$ if we are inside the Edgeworth box, $(3-1)$ and $(3-2)$ imply that

$$10 - x_2^1 = \frac{1}{10 - x_1^2} \qquad (3-3)$$

Since there are 5 units of good 1 in total, $x_1^2 = 5 - x_1^1$, so that $(3-3)$ can be written

$$10 - x_2^1 = \frac{1}{5 + x_1^1} \qquad (3-4)$$

or

$$x_2^1 = 10 - \frac{1}{5 + x_1^1} \qquad (3-5)$$

Equation $(3-5)$ defines an upward–sloping curve in the Edgeworth box, starting on the left edge, at $x_1^1 = 0, x_2^1 = 9.8$, and hitting the right edge at $x_1^1 = 5, x_2^1 = 9.9$.

In addition, there are corner solutions, in which one person's consumption of one good is 0.

The left side of the Edgeworth Box, for values of x_2^1 between 0 and 9.8 is also part of the contract curve. If $x_1^1 = 0, 0 \leq x_2^1 \leq 9.8, x_1^2 = 5, x_2^2 = 20 - x_2^1$, then we have a corner solution : $u_1^1/u_2^1 < u_1^2/u_2^2$. Here person 2 would like to trade some of good 2 to person 1, in exchange for some of good 1, but person 1's consumption of good 1 has already been driven down to 0.

Similarly, the right edge of the Edgeworth Box, for values of x_2^1 between 9.9 and 20 is also part of the contract curve. For example, if $x_1^1 = 5, x_2^1 = 15, x_1^2 = 0, x_2^2 = 5$, then there is no way to make person 1 better off without making person 2 worse off. (The only way to make person 1 better off in this case is to give her more of good 2, since she already has all the available supply of good 1. And giving person 1 more of good 2, with nothing in exchange, must make person 2 worse off.)

note : the fact that the contract curve also includes parts of the sides of the Edgeworth Box (as in the figure below) was worth 5 marks in the grading

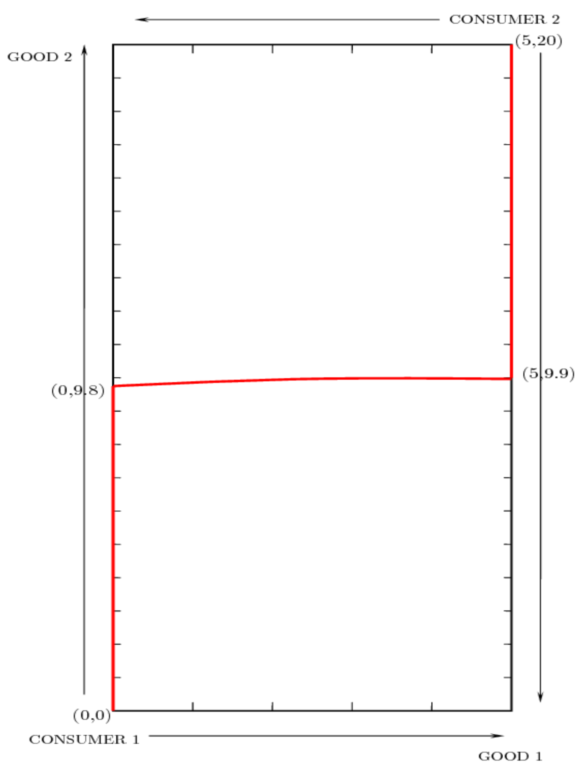

Figure : the contract curve (including parts of the left and right side of the Edgeworth Box)

Q4. What are the allocations of in the core of the following 3–person, 2–good economy? Person i's preferences can be represented by the utility function $u^i(x_1^i, x_2^i)$, where

$$u^1(x_1^1, x_2^1) = x_1^1 + x_2^1$$

$$u^2(x_1^2, x_2^2) = x_1^2 x_2^2$$

$$u^3(x_1^3, x_2^3) = x_1^3 + x_2^3$$

and the endowment vectors of the three people are $\mathbf{e}^1 = (1,1), \mathbf{e}^2 = (2,0), \mathbf{e}^3 = (0,2)$.

A4. Consider first the Pareto–optimal allocations. Since $MRS^1 = 1$, any Pareto–optimal allocation must have $MRS^1 = MRS^2 = MRS^3 = 1$ (unless the allocation is at a corner, with one or more of the x_j^i's equal to 0).

So $MRS^2 = 1$ at any Pareto–optimal allocation, which means that $x_1^2 = x_2^2$, given person 2's Cobb–Douglas preferences.

So the Pareto–optimal allocations are any $\{(x_1^1, x_2^1), (b,b), (x_1^3, x_2^3)$, with $x_1^1 + x_1^3 = 3 - b$, and $x_2^1 + x_2^3 = 3 - b$. The facts that person 1 gets the same quantity of each good, and that aggregate endowments of each good are the same (3), imply that $x_1^1 + x_1^3 = x_2^1 + x_2^3 = 3 - b$.

Consider next the "participation constraint", that any allocation in the core be at least as good for each person as her endowment. Since $u^1(e_1^1, e_2^1) = 1 + 1 = 2$, it therefore must be the case that, for any allocation in the core,

$$x_1^1 + x_2^1 \geq 2 \qquad (4-1)$$

Otherwise person #1 would want to leave. Similarly, $u^3(e_1^3, e_2^3) = 0 + 2 = 2$, so that

$$x_1^3 + x_2^3 \geq 2 \qquad (4-2)$$

in order to induce person #3 to join.

But person #2 gets utility (2)(0) from consuming her endowment vector, so that the participation constraint implies only that $b \geq 0$.

Now suppose next that $x_1^1 + x_2^1 > 2$, so that person #1 actually does better in the coalition of the whole than she does on her own. Because of constraint $(4-2)$, if $x_1^1 + x_2^1 > 2$, then $b < 1$.

In this proposed allocation for the core then, person #1 gets (x_1^1, x_2^1), with $x_1^1 + x_2^1 > 2$, person #2 gets (b,b) with $b < 1$, and person #3 gets (x_1^3, x_2^3) with $x_1^3 + x_2^3 \geq 2$. The assumption that $x_1^1 + x_2^1 > 2$ means that

$$2b + x_1^3 + x_2^3 < 4 \qquad (4-3)$$

since there are 3 units of each good in total.

Consider the following proposal by person #3, to form a coalition with person #2 (and without person #1) : person #2 gets (b,b), and person #3, the proposer, gets what is left from person #2 and person #3's endowments, $(2-b, 2-b)$. From equation $(4-3)$, in the original proposal, person #3 gets $x_1^3 + x_2^3 < 4 - 2b$. So the proposed new coalition, and its allocation $\mathbf{x}^2 = (b,b), \mathbf{x}^3 =$

$(2-b, 2-b)$ makes person #3 strictly better off than she is in the original allocation (in which $x_1^1 + x_2^1 > 2$), and makes person #2 no worse off. Thus the original allocation has been blocked, by a coalition of person #2 and person #3 — because $x_1^1 + x_2^1 > 2$ in the original allocation.

Therefore, if an allocation is in the core, it must be the case that

$$x_1^1 + x_2^1 \leq 2 \tag{4-4}$$

or else it could be blocked by a coalition of person #2 and person #3.

Equations $(4-1)$ and $(4-4)$ together imply that person #1 must get a utility level of 2 (from some bundle with $x_1^1 + x_2^1 = 2$) in any allocation in the core.

Finally, suppose that person #3 were to get utility greater than her "reservation utility" of $e_1^3 + e_2^3 = 2$, from some bundle in which $x_1^3 + x_2^3 > 2$. If $x_1^3 + x_2^3 > 2$, and if $x_1^1 + x_2^1 = 2$, then $b < 1$. Person #2 can do better by proposing a new coalition with person #1 alone : person #1 gets $(2, 0)$ and person #2 gets $(1, 1)$. This allocation is feasible for this 2–person coalition, since that coalition has an aggregate endowment of $\mathbf{e}^1 + \mathbf{e}^2 = (3, 1)$. It gives person #1 the same utility she gets in the original allocation, 2. It gives person #2 higher utility than she gets in the original proposed allocation, 1 instead of $b^2 < 1$. So person #1 and person #2 can block any allocation, if the allocation proposes that $x_1^3 + x_2^3 > 2$. Hence the possibility of blocking by a coalition of person #1 and person #2 means that

$$x_1^2 + x_2^3 \leq 2 \tag{4-5}$$

if an allocation is in the core.

Conditions $(4-2)$ and $(4-5)$ together imply that person #3 must get exactly her reservation utility, 2, if an allocation is in the core.

So the core consists of any allocations for which the following conditions hold :

$$x_1^1 + x_2^1 = 2 \tag{4-6}$$

$$x_1^2 = x_2^2 = 1 \tag{4-7}$$

$$x_1^3 = 2 - x_1^1 \quad ; \quad x_2^3 = 2 - x_2^1 \tag{4-8}$$

Here neither person #1 nor person #3 gain from the ability to trade with the others. But person #2 gains considerably. The presence of the other 2 people enables him, in effect, to play person #1 off against person #3.

Q5. What is the competitive equilibrium in which there are 1 million people of type 1, and 1 million people of type 2, in which each type–1 person has an endowment vector $\mathbf{e}^1 = (7, 1)$ and preferences represented by the utility function

$$u^1(x_1^1, x_2^1) = [x_1^1]^2 x_2^1$$

and each type–2 person has an endowment vector $\mathbf{e}^2 = (5, 1)$ and preferences represented by the utility function

$$u^2(x_1^2, x_2^2) = x_1^2 + \ln(x_2^2) \qquad ?$$

A5. By Walras's Law, to find the equilibrium, it is sufficient to find the relative prices which clear the market for one of the 2 goods.

Also, only relative prices matter, so that good #1 can be chosen as the numéraire.

So consider the demand for good #2.

Type–1 people have Cobb–Douglas preferences, so that the demand function for each type–1 person for good 2 can be written

$$x_2^1 = \frac{y_1}{3p} \tag{5-1}$$

where y_1 is the income of a type–1 person, and p is the relative price of good 2.

Since each type–1 person has an endowment vector $\mathbf{e}^1 = (7, 1)$, y_1 is the value of a type–1 person's endowment, or

$$y_1 = 7 + p \tag{5-2}$$

since good 1 is the numéraire. Substituting from $(5-2)$ into $(5-1)$,

$$x_2^1 = \frac{7+p}{3p} \tag{5-3}$$

Type–2 people have quasi–linear preferences, so that each type–2 person's demand for good #2 is

$$x_2^2 = \frac{1}{p} \tag{5-4}$$

Since there are equal numbers of each type, market clearance for good #2 requires that

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 \tag{5-5}$$

or

$$\frac{7+p}{3p} + \frac{1}{p} = 2 \tag{5-6}$$

which can be written

$$(7+p) + 3 = 6p \tag{5-7}$$

or

$$p = 2$$

So the equilibrium price vector is $\mathbf{p} = k(1, p)$, for any positive constant k.

Substituting $p = 2$ into equations $(5-3)$ and $(5-4)$ implies that $x_2^1 = 1.5$ and $x_2^2 = 0.5$. Because of the Cobb–Douglas preferences of type–1 people, their demand for good #1 is

$$x_1^1 = \frac{2(7+p)}{3} = 6 \tag{5-8}$$

Since each type–1 person consumes one unit less of good #1 than her endowment, in equilibrium it must be true that $x_1^2 = e_1^2 + 1 = 6$, so that the equilibrium consumption vectors are $\mathbf{x}^1 = (6, 1.5)$ and $\mathbf{x}^2 = (6, 0.5)$.