GS/ECON 5010 section "B"' Answers to Assignment 2 October 2011

*Q*1. Could the following 3 equations be Hicksian demand functions (if the reference level of utility u were high enough so that $u + \ln p_2 + \ln p_3 \geq 2 \ln p_1$)? Explain briefly.

$$x_1(\mathbf{p}, u) = u - 2\ln p_1 + \ln p_2 + \ln p_3$$
$$x_2(\mathbf{p}, u) = \frac{p_1}{p_2}$$
$$x_3(\mathbf{p}, u) = \frac{p_1}{p_3}$$

*A*1 Given the proposed Hicksian demand functions, the consumer's expenditure function $e(\mathbf{p}, u)$ would have to equal $p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u) + p_3 x_3^H(\mathbf{p}, u)$, or here

$$e(\mathbf{p}, u) = p_1 u - 2p_1 \ln p_1 + p_1 \ln p_2 + p_1 \ln p_3 + 2p_1 \tag{1-1}$$

Theorem 1.7 in *Jehle and Reny* lists the properties which an expenditure function must have.

It must be increasing in u, which $e(\mathbf{p}, u)$ is. If we calculate the first derivatives of $e(\mathbf{p}, u)$ with respect to the prices,

$$e_1(\mathbf{p}, u) = u - 2\ln p_1 - 2 + \ln p_1 + \ln p_3 + 2 = u - 2\ln p_1 + \ln p_2 + \ln p_3 = x_1^H(\mathbf{p}, u) \tag{1-2}$$

$$e_2(\mathbf{p}, u) = \frac{p_1}{p_2} = x_2^H(\mathbf{p}, u) \tag{1-3}$$

$$e_3(\mathbf{p}, u) = \frac{p_1}{p_3} = x_3^H(\mathbf{p}, u) \tag{1-4}$$

Equations $(1-2)$–$(1-4)$ show that $e(\mathbf{p}, u)$ defined by equation $(1-1)$ is increasing in all prices (if $u + \ln p_2 + \ln p_3 > 2\ln p_1$), and that Shepherd's Lemma holds.

If all prices are increased by some factor k, then

$$e(k\mathbf{p}, u) = kp_1 u - 2kp_1 \ln p_1 + kp_1 \ln p_2 + p_1 k \ln p_3 + 2kp_1 = ke(\mathbf{p}, u) \tag{1-5}$$

when we take into account the property that

$$\ln kp_i = \ln p_i + \ln k$$

so that $e(\mathbf{p}, u)$ is homogeneous of degree 1 in all prices.

The function is also continuous, and equals 0 if u is low enough, and gets arbitrarily large as $u \to \infty$.

So $e(\mathbf{p}, u)$ will satisfy all the properties listed in Proposition 1.7 if it is also a concave function.

To check concavity, differentiate equations $(1-2) - (1-4)$ with respect to the three prices, to get the matrix of second derivatives (of $e(\mathbf{p}, u)$ with respect to prices), which is

$$\begin{pmatrix} -\frac{2}{p_1} & \frac{1}{p_2} & \frac{1}{p_3} \\ \frac{1}{p_2} & -\frac{p_1}{(p_2)^2} & 0 \\ \frac{1}{p_3} & 0 & -\frac{p_1}{(p_3)^2} \end{pmatrix}$$

The determinants of the principal minors of this matrix are : $-2/p_1$ for the 1–by–1 minor, $+p_1/(p_2)^2$ for the 2–by–2 minor, and 0 for the whole matrix. Thus the determinants of the principal minors alternate in sign, so that the matrix of second derivatives of the expenditure function is negative semi–definite, meaning that the expenditure function is concave in prices.

So the three demand functions could be Hickisian demand functions.

(Section 2.1 of the text, not covered in class, shows how to find the direct utility function $u(\mathbf{x})$ corresponding to these demand functions, that is the function $u(\mathbf{x})$ such that minimizing $\mathbf{p} \cdot \mathbf{x}$ subject to the constraint $u(\mathbf{x}) \geq \bar{u}$ generates $e(\mathbf{p}, \bar{u})$ as a solution. Here it is $u(\mathbf{x}) = x_1 + \ln x_2 + \ln x_3$.)

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices p^t of three different commodities at four different times, and the quantities x^t of the 3 goods chosen at the four different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (50, 10, 40)$ when the price vector was $\mathbf{p} = (2, 1, 1)$.)

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	1	1	1	30	40	30
2	2	1	1	50	10	40
3	1	2	1	60	20	10
4	1	1	2	30	50	20

*A*2. One way of finding the violations of the strong and weak axioms of revealed preference is to first construct the matrix, in which the element M_{ij} is the cost of bundle $\mathbf{x}^j$ at prices $\mathbf{p}^i$. Here that matrix is

$$\begin{pmatrix} 100 & 100 & 90 & 100 \\ 130 & 150 & 150 & 130 \\ 140 & 110 & 110 & 150 \\ 130 & 140 & 100 & 120 \end{pmatrix}$$

Using this matrix, the bundle $\mathbf{x}^i$ is directly revealed preferred to the bundle $\mathbf{x}^j$ if $M_{ii} \geq M_{ij}$. For example, row 3 of the matrix has $X_{33} = X_{32}$: that means that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^2$, since bundle $\mathbf{x}^2$ was affordable in period 2 (it cost \$110), and the person instead chose bundle $\mathbf{x}^3$.

So row #1 of the matrix shows that $\mathbf{x}^1$ is directly revealed preferred to each of the other three bundles, since all 4 bundles were affordable in period #1, when the person chose bundle $\mathbf{x}^1$. Similarly row #2 shows that bundle #2 is directly revealed preferred to each of the other 3 bundles.

These first two rows give a violation of $WARP$, since bundle $\mathbf{x}^1$ is directly revealed preferred to bundle $\mathbf{x}^2$ which is itself directly revealed preferred to bundle $\mathbf{x}^1$.

Meanwhile row #3 shows that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^2$, and row #4 shows that bundle $\mathbf{x}^4$ is directly revelaed preferred to $\mathbf{x}^3$.

So there are 2 violations of $WARP$: $\mathbf{x}^1$ compared to $\mathbf{x}^2$ and $\mathbf{x}^2$ compared to $\mathbf{x}^3$.

There are several more violations of $SARP$. In fact, every possible pair of alternatives violate $SARP$. If I use the shorthand "i DRP j" to indicate that $\mathbf{x}^i$ is directly revealed preferred to $\mathbf{x}^j$, then :

— 1 DRP 2 DRP 1
— 1 DRP 3 DRP 2 DRP 1
— 1 DRP 4 DRP 3 DRP 2 DRP 1
— 2 DRP 3 DRP 2
— 2 DRP 4 DRP 3 DRP 2
— 3 DRP 2 DRP 4 DRP 3

*Q*3. If a person was an expected utility maximizer with a utility–of–wealth function

$$u(W) = -W^3 + 30W^2 + 30,000,000W$$

(for $W < 10,000$, where W is her wealth, in thousands of dollars), give an example of a gamble g for which $E[u(g)] < u(Eg)$ for this person, and an example of a gamble g' for which $E[u(g')] > u(Eg')$.

*A*3. This utility–of–wealth is convex when $0 < W < 10$, and concave for $W > 10$, since

$$u'(W) = -3W^2 + 60W + 30000000$$

and

$$u'(W) = -6W + 60$$

The linear term, $30000000W$ is just there to ensure that $u'(W) > 0$ for $W < 10,000$, and does not affect the person's attitude to risk. The figure below shows $u(W) - 30000000W$, which reflects the same attitude towards risk as does $u(W)$.

So, in particular, this person will be a risk lover for any gamble $g = (p_1 \circ W_1, p_2 \circ W_2, \cdots, p_n \circ W_n)$ for which $10 \geq W_1 > W_2 > \cdots > W_n$. An example is the gamble

$$g = (0.5 \circ 10, 0.5 \circ 0)$$

Here

$$Eu(g) = 0.5 * (-10^3 + 30(10^2) + 30000000(10)) = 150001000$$

and

$$Eg = 5$$

Question 3 : Figure

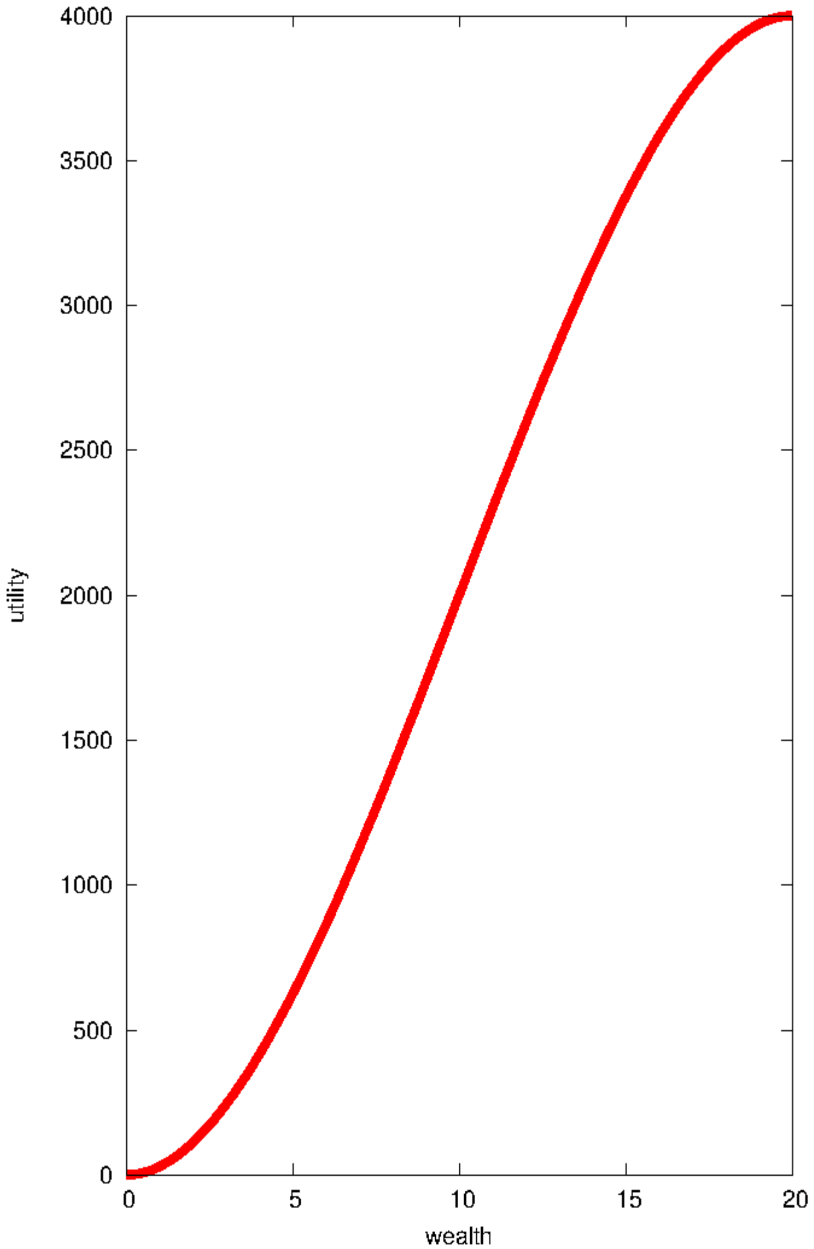

Figure : u(W)-30,000,000W

so that

$$u(Eg) = -5^3 + 30 * 25 + 30000000(5) = 150000625$$

and $u(Eg) < E(u(g))$.

And for any gamble $g = (p_1 \circ W_1, p_2 \circ W_2, \cdots, p_n \circ W_n)$ for which $W_1 > W_2 > \cdots > W_n \geq 10$, the person will be risk averse. An example is the gamble

$$= (0.5 \circ 20, 0.5 \circ 10)$$

Here

$$Eu(g) = 0.5*(-10^3+30(10^2)+30000000(10))+0.5*(-20^3+30(20^2)+30000000(20)) = 450003000$$

and

$$Eg = 15$$

so that

$$u(Eg) = -15^3 + 30(15^2) + 30000000(15) = 450003375$$

and $u(Eg) > Eu(g)$.

*Q*4. How much would a person with wealth W be willing to pay for full insurance against a loss of L, if the probability of the loss were π, and if the person had a constant coefficient of relative risk aversion of 2?

*A*4. If a person has a constant coefficint of relative risk aversion β, then her utility–of–wealth function can be written

$$u(W) = \frac{1}{1-\beta} W^{1-\beta}$$

so that here, with $\beta = 2$, the person's utility–of–wealth function can be written

$$u(W) = -\frac{1}{W}$$

If she has initial wealth W, and expects to suffer a loss of L with probability π, then her expected utility is

$$-(1-\pi)\frac{1}{W} - \pi\frac{1}{W-L} \tag{4-1}$$

On the other hand, if she buys full insurance against that loss, at a price of P, then she will have a certain wealth of $W_0 - P$, and expected utility of

$$-\frac{1}{W-P} \tag{4-2}$$

The most that she would be willing to pay for insurance is the amount P which makes expression $(4-2)$ equal to expression $(4-1)$: for any lower price of insurance, her utility after buying the insurance $((4-2))$ will be larger than her expected utility of taking the risk $((4-1))$.

Solving for P,

$$(1-\pi)\frac{1}{W}+\pi\frac{1}{W-L}=\frac{1}{W-P} \tag{4-3}$$

or

$$\frac{(1-\pi)(W-L)+\pi W}{W(W-L)}=\frac{1}{W-P} \tag{4-4}$$

or

$$\frac{W-(1-\pi)L}{W(W-L)}=\frac{1}{W-P} \tag{4-5}$$

Turning both sides of $(4-5)$ upside down

$$W-P=\frac{W(W-L)}{W-(1-\pi)L} \tag{4-6}$$

or

$$P=W-\frac{W(W-L)}{W-(1-\pi)L} \tag{4-7}$$

which means that

$$P=\pi L\frac{W}{W-(1-\pi)L} \tag{4-8}$$

Equation $(4-8)$ shows that the price this person is willing to pay must exceed the expected loss πL (since she is risk averse). But since

$$\frac{P}{\pi L}=\frac{W}{W-(1-\pi)L} \tag{4-9}$$

if her wealth W is large relative to the loss L, then the price she is willing to pay for the insurance falls, and approaches the expected loss as her wealth grows very large relative to the loss.

*Q*5. For what values of (x_1, x_2) does the production function

$$f(x_1,x_2)=a\sqrt{x_1}+b(x_2)^2$$

exhibit locally increasing returns to scale, where a and b are positive constants?

*A*5. The measure of local returns to scale is $\mu(x_1,x_2)$, defined (in definition 3.4 of the text) by

$$\mu(x_1,x_2)=\frac{f_1(x_1,x_2)x_1+f_2(x_1,x_2)x_2}{f(x_1,x_2)}$$

where f_i denotes the partial derivative with respect to x_i.

Here

$$f_1(x_1,x_2)=\frac{a}{2\sqrt{x_1}}$$

and

$$f_2(x_1,x_2)=2bx_2$$

so that

$$f_1x_1+f_2x_2 = \frac{a}{2}\sqrt{x_1}+2b(x_2)^2 = a\sqrt{x_1}+b(x_2)^2-\frac{a}{2}\sqrt{x_1}+b(x_2)^2 = f(x_1,x_2)-\frac{a}{2}\sqrt{x_1}+b(x_2)^2 \quad (5-1)$$

Equation $(5-1)$ shows that $f_1x_1+f_2x_2 > f(x_1,x_2)$ if and only if

$$b(x_2)^2 > \frac{a}{2}\sqrt{x_1} \quad (5-2)$$

Since $\mu(x_1,x_2) > 1$ if and only if $f_1x_1+f_2x_2 > f(x_1,x_2)$, inequality $(5-2)$ is exactly the condition for $\mu(x_1,x_2)$ to exceed 1.

Re–writing that condition $(5-2)$, this production function will exhibit locally increasing returns to scale if and only if

$$x_1 < \frac{4b^2}{a^2}(x_2)^4 \quad (5-3)$$