GS/ECON 5010 section "B"' Answers to Assignment 2 October 2012

*Q*1. Could the following 3 equations be Hicksian demand functions (if the reference level of utility $u$ were high enough that $u > \ln p_1 - \ln p_2 - \ln p_3$)? Explain briefly.

$$x_1(\mathbf{p}, u) = u - \ln p_1 + \ln p_2 + \ln p_3$$

$$x_2(\mathbf{p}, u) = \frac{p_1}{p_2}$$

$$x_3(\mathbf{p}, u) = \frac{p_1}{p_3}$$

*A*1 Given the proposed Hicksian demand functions, the consumer's expenditure function $e(\mathbf{p}, u)$ would have to equal $p_1 x_1^H(\mathbf{p}, u) + p_2 x_2^H(\mathbf{p}, u) + p_3 x_3^H(\mathbf{p}, u)$, or here

$$e(\mathbf{p}, u) = p_1 u - p_1 \ln p_1 + p_1 \ln p_2 + p_1 \ln p_3 + 2p_1 \tag{1-1}$$

Theorem 1.7 in *Jehle and Reny* lists the properties which an expenditure function must have.

It must be increasing in $u$, which $e(\mathbf{p}, u)$ is. If we calculate the first derivatives of $e(\mathbf{p}, u)$ with respect to the prices,

$$e_1(\mathbf{p}, u) = u - \ln p_1 - 1 + \ln p_2 + \ln p_3 + 2 = u - \ln p_1 + \ln p_2 + \ln p_3 + 1 = x_1^H(\mathbf{p}, u) + 1 \tag{1-2}$$

$$e_2(\mathbf{p}, u) = \frac{p_1}{p_2} = x_2^H(\mathbf{p}, u) \tag{1-3}$$

$$e_3(\mathbf{p}, u) = \frac{p_1}{p_3} = x_3^H(\mathbf{p}, u) \tag{1-4}$$

Equations $(1-2)$–$(1-4)$ show that $e(\mathbf{p}, u)$ defined by equation $(1-1)$ is increasing in all prices. But Shepherd's Lemma does **NOT** hold : $e_1(\mathbf{p}, u) = x_1^H(\mathbf{p}, u) + 1 > x_1^H(\mathbf{p}, u)$

The expenditure function can also be written

$$e(\mathbf{p}, u) = p_1 u - p_1 \ln\left(\frac{p_1}{p_2}\right) + p_1 \ln p_3 + 2p_1 \tag{1-5}$$

Equation $(1-5)$ shows that the expenditure function here is **NOT** homogeneous of degree 1 in prices together. If we double all prices, 3 of the 4 terms on the right–hand side of $(1-5)$ will double. But the third term $(p_1 \ln p_3$ would more than double, to $2p_1[\ln(p_3) + \ln(2)]$ if all prices doubled.

More simply : the alleged Hicksian demand function $x_1(\mathbf{p}, u) = u - \ln p_1 + \ln p_2 + \ln p_3$ is also not homogeneous of degree 0 in prices. It can be written

$$x_1(\mathbf{p}, u) = u + \ln\left(\frac{p_2}{p_1}\right) + \ln p_3 \tag{1-6}$$

so that doubling all prices would actually increase $x_1^H$, which means it cannot be a Hicksian demand function.

So the properties of Theorem 1.7 do not hold for the expenditure function constructed from the (alleged) Hicksian demand functions given in the question : the three functions $x_1^H(\mathbf{p}, u), x_2^H(\mathbf{p}, u), x_3^H(\mathbf{p}, u)$ cannot be Hicksian demand functions. (Subtract $2 \ln p_1$ instead of just $\ln p_1$ on the right side of the definition of $x_1(\mathbf{p}, u)$, and we will have Hicksian demand functions, as shown in question #1 of the Fall 2011 Assignment #2.)

*Q*2. Find all the violations of the strong and weak axioms of revealed preference in the following table, which indicates the prices $p^t$ of three different commodities at four different times, and the quantities $x^t$ of the 3 goods chosen at the four different times. (For example, the second row indicates that the consumer chose the bundle $\mathbf{x} = (30, 40, 30)$ when the price vector was $\mathbf{p} = (2, 1, 2)$.)

| $t$ | $p_1^t$ | $p_2^t$ | $p_3^t$ | $x_1^t$ | $x_2^t$ | $x_3^t$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 50 | 20 | 30 |
| 2 | 2 | 1 | 2 | 30 | 40 | 30 |
| 3 | 3 | 2 | 2 | 60 | 30 | 8 |
| 4 | 2 | 2 | 1 | 50 | 40 | 20 |

*A*2. One way of finding the violations of the strong and weak axioms of revealed preference is first to construct the matrix, in which the element $M_{ij}$ is the cost of bundle $\mathbf{x}^j$ at prices $\mathbf{p}^i$. Here that matrix is

$$\begin{pmatrix} 200 & 200 & 196 & 220 \\ 180 & 160 & 166 & 180 \\ 250 & 230 & 256 & 270 \\ 170 & 170 & 188 & 200 \end{pmatrix}$$

Using this matrix, the bundle $\mathbf{x}^i$ is directly revealed preferred to the bundle $\mathbf{x}^j$ if $M_{ii} \geq M_{ij}$. For example, row 3 of the matrix has $X_{33} > X_{32}$ : that means that bundle $\mathbf{x}^3$ is directly revealed preferred to bundle $\mathbf{x}^2$, since bundle $\mathbf{x}^2$ was affordable in period 3 (it cost \$230), and the person instead chose bundle $\mathbf{x}^3$.

The second row of the table shows that the bundle $\mathbf{x}^2$ is not (directly) revealed preferred to any other bundle. That means that we cannot have any violations of $WARP$ or $SARP$ involving the bundle $\mathbf{x}^2$ : to be part of a chain of "is revealed directly preferred to" in a violation of $SARP$, a bundle must be revealed preferred to something else.

In the top row, bundle $\mathbf{x}^1$ is directly revealed preferred to bundles $\mathbf{x}^2$ and $\mathbf{x}^3$. Row 3 then provides a violation of $WARP$ : row 3 shows that $\mathbf{x}^3$ is directly revealed preferred to $\mathbf{x}^1$, and row 1 shows that $\mathbf{x}^1$ is directly revealed preferred to $\mathbf{x}^3$.

Row 4 shows that bundle $\mathbf{x}^4$ is directly revealed preferred to each of the other three bundles. But no other bundle is directly revealed preferred to $\mathbf{x}^4$. That is, the cost total in the fourth column is bigger than the cost total on the diagonal in each of rows 1,2 and 3.

So there is one violation of $WARP$ here, and that's also the only violation of $SARP$ : $\mathbf{x}^3$ directly revealed preferred to $\mathbf{x}^1$, and $\mathbf{x}^1$ directly revealed preferred to $\mathbf{x}^3$.

*Q*3. If a person was an expected utility maximizer with a utility–of–wealth function

$$u(W) = W^2 - \frac{8000000}{W}$$

(where $W$ is her wealth, in thousands of dollars), give an example of a gamble $g$ for which $E[u(g)] < u(Eg)$ for this person, and an example of a gamble $g'$ for which $E[u(g')] > u(Eg')$.

*A*3. This utility–of–wealth is concave when $0 < W < 200$, and convex for $W > 200$, since

$$u'(W) = 2W + \frac{8000000}{W^2} \qquad (3-1)$$

and

$$u''(W) = 2 - \frac{16000000}{W^3} \qquad (3-2)$$

From equation $(3-2)$, $u''(W) < 0$ if and only if

$$W^3 < 8000000 \qquad (3-3)$$

Since $(200)^3 = 8000000$, $u''(W) < 0$ if and only if $W < 200$.

So, in particular, this person will be risk averse for any gamble $g = (p_1 \circ W_1, p_2 \circ W_2, \cdots, p_n \circ W_n)$ for which $200 \geq W_1 > W_2 > \cdots > W_n$. An example is the gamble

$$g = (0.5 \circ 5, 0.5 \circ 1)$$

Here

$$Eu(g) = (0.5)(25 - 800000) + (0.5)(1 - 4000000) = 13 - 2,200,000 = -2,199,987$$

and

$$Eg = 3$$

so that

$$u(Eg) = 9 - \frac{4000000}{3} = -1,333,324.33$$

and $u(Eg) > E(u(g))$.

And for any gamble $g = (p_1 \circ W_1, p_2 \circ W_2, \cdots, p_n \circ W_n)$ for which $W_1 > W_2 > \cdots > W_n \geq 200$, the person will be a risk lover. An example is the gamble

$$= (0.5 \circ 3000, 0.5 \circ 1000)$$

Here

$$Eu(g) = (0.5)(9000000 - \frac{8000000}{3000}) + (0.5)(1000000 - \frac{8000000}{1000}) = 5000000 - 10666.67 = 5989933.33$$

and

$$Eg = 2000$$

so that

$$u(Eg) = 4000000 - \frac{8000000}{2000} = 3996000$$

and $u(Eg) < Eu(g)$.

*Q*4. How much insurance would a person buy against a loss of $L$ dollars, if the person had initial wealth of $W > L$, if the probability of the loss were $\pi$, and if the price of a dollar of insurance coverage were $p$ dollars (with $p \geq \pi$), and if the person had a constant coefficient of relative risk aversion of $\beta > 0$?

*A*4. If a person has a constant coefficient of relative risk aversion $\beta$, then her utility–of–wealth function can be written

$$u(W) = \frac{1}{1-\beta} W^{1-\beta} \tag{4-1}$$

If she has initial wealth $W$, expects to suffer a loss of $L$ with probability $\pi$, and buys $X$ dollars worth of coverage, at a cost of $p$ per dollar of coverage, then her wealth will be

$$W_g = W - pX \tag{4-3}$$

in the good state (in which she does not suffer a loss) and

$$W_b = W - L - pX + X \tag{4-3}$$

in the bad state (in which she suffers a loss, and gets a payment of $X$ from the insurance company as compensation).

So her expected utility is

$$EU = (1-\pi)u(W_g) + \pi U(W_b) \tag{4-4}$$

She wants to pick a level of coverage $X$ so as to maximize her expected utility : so she tries to find the value of $X$ for which the derivative of $EU$ with respect to $X$ is zero.

From equations $(4-1)$–$(4-4)$,

$$\frac{\partial EU}{\partial X} = -p(1-\pi)[W - pX]^{-\beta} + (1-p)\pi[W - L + (1-p)X]^{-\beta} = 0 \tag{4-5}$$

Equation $(4-5)$ can be written

$$W - pX = \gamma[W - L + (1-p)X] \tag{4-6}$$

where

$$\gamma = [\frac{p}{1-p}\frac{1-\pi}{\pi}]^{1/\beta} \tag{4-7}$$

Notice that $\gamma = 1$ if $p = \pi$, $\gamma > 1$ if $p > \pi$, and that $\gamma$ is a decreasing function of the degree of relative risk aversion $\beta$ when $p > \pi$.

Equation $(4-6)$ can be re–arranged into

$$X[\gamma - (\gamma - 1)p] = \gamma L - (\gamma - 1)W \tag{4-8}$$

or

$$X = \frac{\gamma}{\gamma - (\gamma - 1)p}L - \frac{\gamma - 1}{\gamma - (\gamma - 1)p}W \tag{4-9}$$

If insurance is actually fair, then $p = \pi$, so that $\gamma = 1$, and equation $(4-9)$ says that the person buys full insurance : $X = L$. [This is true regardless of the value of the person's coefficient of relative risk aversion, since $\gamma = 1$ when $p = \pi$, regardless of the value of $\beta$.]

Equation $(4-9)$ can be re–written as

$$X = L - \frac{\gamma - 1}{\gamma - (\gamma - 1)p}[W - pL] \tag{4-10}$$

Equation $(4-10)$ shows that the person buys less–than–full insurance when $p > \pi$ (so that $\gamma > 1$) ; it shows that the total amount of insurance falls as her wealth increases (holding c onstant the amount $L$ of the loss) ; and the amount of insurance increases as her coefficient of relative risk aversion $\beta$ increases [the fraction $\frac{\gamma-1}{\gamma-(\gamma-1)p}$ in equation $(4-10)$ is an increasing function of $\gamma$, so that the amount of insurance purchased decreases with $\gamma$, and it was stated above that $\gamma$ is a decreasing function of $\beta$ when $p > \pi$.]

Equation $(4-11)$ can also be written

$$\frac{X}{L} = 1 - \frac{\gamma - 1}{\gamma - (\gamma - 1)p}[\frac{W}{L} - p] \tag{4-11}$$

so that the percentage of the loss which a person chose to insure would not vary with her wealth, if the loss were a constant fraction of her wealth. (That is : if $W$ and $L$ both doubled, then her preferred amount of coverage $X$ would also double.)

*Q*5. For what values of $(x_1, x_2, x_3)$ does the production function

$$f(x_1, x_2, x_3) = x_1x_2 + 10\frac{x_3}{x_3 + 1}$$

exhibit locally increasing returns to scale?

*A*5. The measure of local returns to scale is $\mu(x_1, x_2)$, defined (in definition 3.4 of the text) by

$$\mu(\mathbf{x}) = \frac{f_1(\mathbf{x})x_1 + f_2(\mathbf{x})x_2 + f_3(\mathbf{x})x_3}{f(\mathbf{x})}$$

where $f_i$ denotes the partial derivative with respect to $x_i$.

Here

$$f_1(\mathbf{x}) = x_2$$

$$f_2(\mathbf{x}) = x_1$$

and

$$f_3(\mathbf{x}) = \frac{10}{(x_3+1)^2}$$

so that

$$f_1x_1 + f_2x_2 + f_3x_3 = 2x_1x_2 + \frac{10x_3}{(x_3+1)^2} \qquad (5-1)$$

Equation $(5-1)$ shows that $f_1x_1 + f_2x_2 + f_3x_3 > f(x_1, x_2, x_3)$ if and only if

$$x_1x_2 > 10[\frac{x_3}{x_3+1}]^2 \qquad (5-2)$$

Since $\mu(x_1, x_2, x-3) > 1$ if and only if $f_1x_1 + f_2x_2 + f_3x_3 > f(x_1, x_2, x_3)$, inequality $(5-2)$ is exactly the condition for $\mu(x_1, x_2, x_3)$ to exceed 1.

And the production function exhibits locally increasing returns to scale if and only if $\mu(x) > 1$, so that inequality $(5-2)$ is the condition for locally increasing returns to scale.