GS/ECON 5010 section "B"' Answers to Assignment 3 November 2012

$Q1$. What is the profit function, and the long–run supply function, for a perfectly competitive firm with a production function

$$f(x_1, x_2) = \ln x_1 + \ln x_2 - \ln(x_1 + x_2) \qquad ?$$

$A1$ If the firm minimizes the cost of producing a given level of output y, then it must be the case that

$$\frac{f_1}{f_2} = \frac{w_1}{w_2} \tag{1-1}$$

where f_i is the marginal product of input i and w_i is the price (per unit) of input i. For this production function,

$$f_1(x_1, x_2) = \frac{1}{x_1} - \frac{1}{x_1 + x_2} = \frac{x_2}{x_1(x_1 + x_2)} \tag{1-2}$$

$$f_2(x_1, x_2) = \frac{1}{x_2} - \frac{1}{x_1 + x_2} = \frac{x_1}{x_2(x_1 + x_2)} \tag{1-3}$$

so that cost minimization (equation $(1-1)$) implies that

$$[\frac{x_1}{x_2}]^2 = \frac{w_2}{w_1} \tag{1-4}$$

or

$$x_2 = \sqrt{\frac{w_1}{w_2}} x_1 \tag{1-5}$$

and

$$x_1 + x_2 = \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_2}} x_1 \tag{1-6}$$

From the properties of the natural logarithm, the firm's production function can also be written

$$f(x_1, x_2) = \ln[\frac{x_1 x_2}{x_1 + x_2}] \tag{1-7}$$

so that if the output produced is $y = f(x_1, x_2)$, then

$$e^y = \frac{x_1 x_2}{x_1 + x_2} \tag{1-8}$$

Substituting from $(1-5)$ and $(1-6)$ into $(1-8)$,

$$e^y = x_1 \frac{\sqrt{w_1}}{\sqrt{w_1} + \sqrt{w_2}} \tag{1-9}$$

or

$$x_1 = e^y \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_1}} \tag{1-10}$$

Substituting from $(1-10)$ into $(1-5)$,

$$x_2 = e^y \frac{\sqrt{w_1}+\sqrt{w_2}}{\sqrt{w_2}} \tag{1-11}$$

Equations $(1-10)$ and $(1-11)$ are the conditional input demand functions, expressing the quantities of the inputs as functions of the quantity y of the output, and of the prices of the two inputs. Since $C(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y)$, the firm's (long–run total) cost function is

$$C(w_1, w_2, y) = e^y[\sqrt{w_1}+\sqrt{w_2}]^2 \tag{1-12}$$

A competitive firm's long–run profit maximization function is to choose the output level y which maximizes $py - C(w_1, w_2, y)$, with first–order condition $p = C_y(w_1, w_2, y)$. Here, from equation $(1-12)$, that condition is

$$p = e^y[\sqrt{w_1}+\sqrt{w_2}]^2 \tag{1-13}$$

(since the derivative of e^y with respect to y is e^y). Equation $(1-14)$ implies that

$$e^y = p[\sqrt{w_1}+\sqrt{w_2}]^{-2} \tag{1-14}$$

or

$$y = \ln p - 2\ln[\sqrt{w_1}+\sqrt{w_2}] \tag{1-15}$$

Equation $(1-15)$ is the firm's (long–run) supply function $y(p, w_1, w_2)$. From $(1-15)$, when the firm maximizes profit,

$$e^y = \frac{p}{(\sqrt{w_1}+\sqrt{w_2})^2} \tag{1-16}$$

so that

$$C(w_1, w_2, y(p, w_1, w_2)) = p$$

when the firm maximizes profits. Its profit function is $py(p, w_1, w_2) - C(w_1, w_2, y(p, w_1, w_2))$, or

$$\pi(p, w_1, w_2) = p[\ln p - 1 - 2\ln(\sqrt{w_1}+\sqrt{w_2})] \tag{1-17}$$

Differentiation of $(1-17)$ with respect to p yields $(1-15)$, and differentiation of $(1-17)$ with respect to w_i yields the negatives of the unconditional input demand functions

$$-x_1(p, w_1, w_2) = -x_1(w_1, w_2, y(p, w_1, w_2)) = -\frac{p}{\sqrt{w_1}(\sqrt{w_1}+\sqrt{w_2})} \tag{1-18}$$

$$-x_2(p, w_1, w_2) = -x_2(w_1, w_2, y(p, w_1, w_2)) = -\frac{p}{\sqrt{w_2}(\sqrt{w_1}+\sqrt{w_2})} \tag{1-19}$$

$Q2$. What is the equation of the long–run supply curve for a perfectly–competitive industry, in which each of the (many) identical firms has a long run total cost function

$$TC(q) = q^3 - 18q^2 + 111q$$

where q is the quantity of output produced by the firm?

$A2$. Since the long–run total cost function is

$$TC(q) = q^3 - 18q^2 + 111q$$

then a firm's long–run marginal cost and average cost functions are

$$MC(q) = TC'(q) = 3q^2 - 36q + 111 \tag{2-1}$$

$$AC(q) = \frac{TC(q)}{q} = q^2 - 18q + 111 \tag{2-2}$$

Differentiating yet again,

$$MC'(q) = 6q - 36 \tag{2-3}$$

$$AC'(q) = 2q - 18 \tag{2-4}$$

From equations $(2-3)$ and $(2-4)$ both the marginal and average cost curves are U–shaped, with minima at $q = 6$ and $q = 9$ respectively. When $q = 9$,

$$MC(q) = 3(9^2) - 36(9) + 111 = 30 \tag{2-5}$$

$$AC(q) = 81 - 18(9) + 111 = 30 \tag{2-6}$$

confirming that the AC and MC curves cross at the bottom of the AC curve.

With identical firms in perfect competition, in the long run it must be the case that $p = MC$ if firms each maximize their profits, and that $p = AC$ if there is free entry and exit. The only quantity q for which $MC = AC$ is the bottom of each firm's AC curve, $q = 9$.

Thus, in the long–run, the price must be 30, and each firm in the industry must produce 9 units of output. The industry long–run curve is horizontal, at a height of $p = 30$.

*Q*3. Suppose that consumers' preferences could be represented by the utility function

$$u(x_1, x_2) = x_1 + Ax_2 - (0.5)(x_2)^2$$

where A is some positive constant.

Suppose as well that good 1 is provided competitively, at a price of 1.

Good 2 is provided by a monopoly. The monopoly is thinking of the following price policy : customers have to pay a flat fee F in order to be able to buy from the monopoly at all ; they then can buy as much or as little of the monopoly's output as they want, at a price of p per unit. (That is, customers must make an up–front payment of F in order to buy anything at all from the monopoly.)

(i) What is the highest fee F that the monopoly can charge a customer, as a function of the price p per unit which it is charging?

(ii) If the monopoly's production cost is a constant c per unit, what price p, and what fee F should it charge to maximize profits, if all consumers are identical?

*A*3. For this consumer, the marginal utilities of consumption of the two goods are

$$MU_1 = 1 \tag{3-1}$$

$$MU_2 = A - x_2 \tag{3-2}$$

so that her first–order condition for optimality of consumption, $MU_2/MU_1 = p$, where p is the price the monpoly charges, implies that

$$A - x_2 = p \tag{3-3}$$

so that her Marshallian demand function for the output of the monopoly – should she choose to buy anything from the monopoly – is

$$x_2 = A - p \tag{3-4}$$

But she can only buy from the monopoly if she pays the fee F. So she has two choices. She can choose to pay the fee. In this case, she'll buy $A-p$ units of good 2, and have $y - F - p(A-p)$ left to spend on good 1, if y is her income. So $x_1 = y - F - p(A-p)$ and $x_2 = (A-p)$ and her utility is

$$v(y - F, p) = y - F - p(A-p) + A(A-p) - (0.5)(A-p)^2 \tag{3-5}$$

or

$$v(y - F, p) = y - F + (0.5)(A-p)^2 \tag{3-6}$$

On the other hand, if she refuses to pay the fee F, then she can't buy good 2. Then she'll spend all her money on good 1, have $x_1 = y$, and have utility of

$$v(y, \infty) = y \tag{3-7}$$

So she'll be willing to pay the fee F only if $y - F + (0.5)(A-p)^2 > y$, or

$$F \leq (0.5)(A-p)^2 \qquad (3-8)$$

The right–hand side of $(3-8)$ is the highest fee the monopoly can charge, and still have the customer willing to pay it. Notice that the higher is the price, the lower will be the fee the monopoly can charge. (This is true only if $p < A$, but if $p > A$, then the consumer would not want to buy any of the monpoly's good.)

Treating $(3-8)$ as an equality, the monopoly's profit, if it charges a price p, and a fee F satisfying $(3-8)$ will be

$$\pi = (A-p)(p-c) + (0.5)(A-p)^2 \qquad (3-9)$$

The first term in $(3-9)$ is the monopoly's profit from sales : its total sales $A-p$ times the profit per unit of $p-c$. The second term is the revenue it gets from the fee.

Maximization of the monopoly's profit means maximizing $(3-9)$ with respect to p.

$$\pi'(p) = (A-p) - (p-c) - (A-p) \qquad (3-10)$$

so that the price which maximizes $(3-9)$ is $p = c$.

With identical consumer's, this monopoly's most profitable **two–part tariff** is to set price equal to marginal cost : $p = c$. It makes all its profit from the fixed entry fee F, which here equals $(0.5)(A-c)^2$, which is the entirety of the consumer surplus the customer would get from being able to buy at a price equal to the marginal cost.

With identical consumers, the ability to charge an entry fee means that the monopoly can do as well as if it could price discriminate perfectly. It also will choose to provide the same level of output as a competitive industry (but to grab all the benefit from consumers).

[If consumers had different values for A, but the monopoly had to charge the same fee F to all of them, then matters would be very different. The monopoly would charge a price above marginal cost, as it cannot extract all the consumer surplus from all the customers, unless it can charge person–specific fixed fees.]

*Q*4. Another model of duopoly is that of **von Stackelberg**, in which firms choose output levels **sequentially**. That is, firm 1 chooses its output. Firm 2 observes what output level firm 1 has chosen, and then chooses its own output level. What output levels would the 2 firms choose, if they behaved in this manner, if they both produced an identical product for which the market inverse demand function had the equation

$$p = 15 - (q_1 + q_2)$$

if each firm had a total cost function

$$TC = \begin{array}{rl} 1 + 3q_i & \text{if} \quad q_i > 0 \\ 0 & \text{if} \quad q_i = 0 \end{array}$$

where q_i is the output level of firm i? [That is, each firm has a fixed cost of 1, and marginal cost of 3, and the fixed cost can be avoided only if the firm produces nothing at all.]

*A*4. This problem must be solved backwards. First, what is firm 2's reaction to firm 1 producing an output level of q_1? If $q_2 > 0$, then

$$\pi_2 = pq_2 - TC(q_2) = [15 - (q_1 + q_2)]q_2 - (1 + 3q_2) \tag{4-1}$$

Maximizing π_2 with respect to q_2 yields the first–order condition

$$15 - q_1 - 2q_2 - 3 = 0 \tag{4-2}$$

or

$$q_2 = 6 - \frac{q_1}{2} \tag{4-3}$$

But firm 2 will choose to produce a positive level of output only if it earns a positive profit. What is its profit if firm 1 has chosen an output level of q_1, and if firm 2 has responded by choosing $q_2 = 6 - (q_1/2)$? In this case the price is $15 - q_1 - q_2$, which means that

$$p = 15 - q_1 - (6 - \frac{q_1}{2}) = 9 - \frac{q_1}{2} \tag{4-4}$$

Substituting back into $(4-1)$,

$$\pi_2 = [6 - \frac{q_1}{2}][p - 3] - 1 = [6 - \frac{q_1}{2}]^2 - 1 \tag{4-5}$$

So firm 2 can earn a positive profit only if

$$[6 - \frac{q_1}{2}]^2 > 1$$

which is the same thing as

$$6 - \frac{q_1}{2} > 1$$

or

$$q_1 < 10$$

So if $q_1 \geq 10$, then firm 2's best response is to produce nothing at all, since the fixed costs (of 1) imply that it would lose money at any positive level of production. If $q_1 < 10$, firm 2 should produce the output level defined by equation $(4-3)$.

Now consider firm 1's decision. It knows that if it produces an output level of $q_1 < 10$, then firm 2 will follow by producing $6 - \frac{q_1}{2}$, resulting in a price of $9 - \frac{q_1}{2}$. So firm 1's profit, if it chooses an output level of q_1 initially, will be

$$\pi_1 = pq_1 - TC(q_1) = [9 - \frac{q_1}{2}]q_1 - 3q - 1 - 1 \tag{4-6}$$

Maximizing $(4-6)$ with respect to q_1 yields the first–order condition

$$q_1 = 6$$

resulting in profits of

$$\pi_1 = [6 - \frac{6}{2}]10 - 3(6) - 1 = 17$$

On the other hand, if firm 1 produces an output of 10 or more, then firm 2 will shut down completely. That would result in a price of $15 - q_1$, and a profit to firm 1 of

$$(15 - q_1)q_1 - 3q_1 - 1 = 12q_1 - q_1^2 - 1 \tag{4-7}$$

The expression $(4-7)$ is decreasing in q_1 when $q_1 \geq 6$. That means that, if firm 1 were to find it profitable to have $q_1 \geq 10$, that $q_1 = 10$ would be the best level of output to choose. That level is the smallest level of output for firm 1 which will induce firm 2 to shut down completely.

At $q_1 = 10$ (and $q_2 = 0$), equation $(4-7)$ shows that

$$\pi_1 = 12(10) - 100 - 1 = 19$$

Since $19 > 17$, then the best strategy for firm 1 is to produce an output just high enough that firm 2 cannot make a profit. In other words, firm 1 commits to an output just high enough ($q_1 = 10$) that firm 2 will choose not to enter, because the market is not profitable enough for firm 2 to cover its fixed costs. The Stackelberg equilibrium here has $q_1 = 10$ and $q_2 = 0$.

*Q*5. What does the contract curve look like for a 2–person, 2–good exchange economy, with a total endowment of 20 units of good 1 and 20 units of good 2, if the preferences of the two people could be represented by the utility functions

$$u^1(x_1^1, x_2^1) = x_1^1 + 2x_2^1$$

$$u^2(x_1^2, x_2^2) = 10 - \frac{2}{x_1^2} - \frac{1}{x_2^2}$$

where x_j^i is person i's consumption of good j?

*A*5. The marginal rates of substitution of the two people are

$$MRS^1 = \frac{MU_2^1}{MU_1^1} = 2 \tag{5-1}$$

$$MRS^2 = \frac{MU_2^2}{MU_1^2} = \frac{(x_1^2)^2}{2(x_2^2)^2} \tag{5-2}$$

Since the first person regards the two goods as perfect substitutes, her MRS does not depend on her consumption bundle.

So an allocation inside the Edgeworth box will be efficient only if $MRS^2 = 2$, or if

$$\frac{(x_1^2)^2}{(x_2^2))^2} = 4 \tag{5-3}$$

which is the same as

$$x_1^2 = 2x_2^2 \tag{5-4}$$

That's a straight line in the Edgeworth box. Since $x_1^2 = 20 - x_1^1$ and $x_2^2 = 20 - x_2^1$, equation $(5-4)$ can be written in terms of person 1's consumption as

$$x_2^1 = 10 + \frac{x_1^1}{2} \tag{5-5}$$

Equation $(5-5)$ defines an upward–sloping line in the Edgeworth box, running from the point $(0, 10)$ along the left edge of the box, to the top right–hand corner $(20, 20)$ of the box.

But there also are efficient allocations on the edge of the box, since equation $(5-5)$ does not go through the bottom–left corner of the box. Any allocation in which $x_1^1 = 0$ and $x_2^1 \leq 10$, on the left edge of the box, is also Pareto–optimal. For these allocations, $MRS^1 = 2$, and $MRS^2 \leq 2$. Person 2 would like to trade some of his allotment of good 2 to person 1, in exchange for some more of good 1. But person 1 has no more of good 1 to trade.

Therefore, the contract curve here goes up the left side of the Edgeworth box, from $(0, 0)$ to $(0, 10)$, and then along the line defined by equation $(5-5)$.