GS/ECON 5010 section "B"' Answers to Assignment 2 F2014

*Q*1. For what values of α, β and γ would the following pair of functions represent the Marshallian demand functions of a consumer with income y, facing prices (p_1, p_2)?

$$x_1^M(p_1, p_2, y) = \frac{y}{2p_1} + \frac{p_2}{p_1}$$

$$x_2^M(p_1, p_2, y) = \alpha\frac{y}{p_2} + \beta\frac{p_1}{p_2} - \gamma$$

*A*1. Marshallian demand functions must obey the adding–up constraint, that

$$p_1 x_1^M(p_1, p_2, y) + p_2 x_2^M(p_1, p_2, y) = y \qquad (1-1)$$

for every price–income combination (p_1, p_2, y).

Here

$$p_1 x_1^M(p_1, p_2, y) + p_2 x_2^M(p_1, p_2, y) = \frac{1}{2}y + p_2 + \alpha y + \beta p_1 - \gamma p_2 \qquad (1-2)$$

The only way that the right hand expression in $(1-2)$ can equal y exactly, for any values of p_1, p_2 and y, is if

$$\alpha = \frac{1}{2} \qquad (1-3)$$

$$\beta = 0 \qquad (1-4)$$

$$\gamma = 1 \qquad (1-5)$$

So the "candidate" functions

$$x_1^M(p_1, p_2, y) = \frac{y}{2p_1} + \frac{p_2}{p_1} \qquad (1-6)$$

$$x_2^M(p_1, p_2, y) = \frac{y}{2p_2} - 1 \qquad (1-7)$$

represent Marshallian demand functions if the 2–by–2 Slutsky matrix, with typical element

$$S_{ij} \equiv \frac{\partial x_i^M}{\partial p_j} + x_j^M(p_1, p_2, y)\frac{\partial x_i^M}{\partial y} \qquad (1-8)$$

is symmetric, and negative semi–definite.

From equations $(1-6)$–$(1-8)$, that matrix is

$$S = \begin{pmatrix} -\frac{y}{(p_1)^2} - \frac{p_2}{2(p_1)^2} & \frac{1}{2p_1} + \frac{y}{4p_1p_2} \\ \frac{1}{2p_1} + \frac{y}{4p_1p_2} & -\frac{y}{4(p_2)^2} - \frac{1}{2p_2} \end{pmatrix} \qquad (1-9)$$

So the matrix is symmetric. The elements on the diagonal are negative. And the determinant of the matrix in equation $(1-9)$ is :

$$\frac{y^2}{16(p_1)^2(p_2)^2} + \frac{y}{16(p_1)^2 p_2} + \frac{1}{(p_1)^2} - \frac{y^2}{16(p_1)^2(p_2)^2} - \frac{y}{16(p_1)^2 p_2} - \frac{1}{(p_1)^2} = 0$$

so that the matrix is negative semi–definite, since the elements on the diagonal are negative and the determinant is 0.

*Q*2. The following table lists the prices of 3 goods, and the quantities a consumer chose of the goods, in 3 different years.

For what values of A do these data satisfy the strong axiom of revealed preference?

t	p_1^t	p_2^t	p_3^t	x_1^t	x_2^t	x_3^t
1	5	1	1	25	20	25
2	5	1	1	25	40	40
3	1	1	5	25	30	A

*A*2. The costs of the three chosen bundles, in each of the three periods, can be represented by the matrix below, where element ij is the price of bundle j in year i :

$$\begin{matrix} 170 & 205 & 155+A \\ 170 & 205 & 155+A \\ 170 & 265 & 55+5A \end{matrix}$$

Since the bundle $\mathbf{x}^1$ is not directly revealed preferred to the bundle $\mathbf{x}^2$, SARP can only be violated if the bundle $\mathbf{x}^3$ is directly preferred to either of the other bundles. So if $A < 23$, the third row of the matrix shows that $\mathbf{x}^3$ is not revealed preferred to either of the other bundles. In that case there can be no violation of SARP.

If $23 \leq A < 42$ then $\mathbf{x}^3$ is revealed directly to be preferred to $\mathbf{x}^1$, but not to $\mathbf{x}^2$. In that case, the cost of bundle $\mathbf{x}^3$ in year 1 falls between 178 and 197, which is greater than the cost of the bundle $\mathbf{x}^1$, which was actually chosen in the year. So again,no possible violations of SARP : $\mathbf{x}^1$ is not revealed directly to be preferred to anything, and $\mathbf{x}^3$ is not revealed preferred to $\mathbf{x}^2$.

If $A \geq 42$, then bundle $\mathbf{x}^3$ is revealed preferred to bundle $\mathbf{x}^2$. So there is a violation of $WARP$ if bundle $\mathbf{x}^2$ is revealed directly to be preferred to bundle $\mathbf{x}^3$. This will be the case if and only if $A \leq 50$.

So violations of $SARP$ can occur only if and only if $42 \leq A \leq 50$.

*Q*3. Write down a von Neumann–Morgenstern expected utility function for a person who would be willing to choose **each** of the following actions if her initial wealth were $W = 200$:

(*i*) Pay more than \$100 for insurance against a disaster which would lose her all her wealth with probability 0.5.

(*ii*) Pay more than \$100 for an investment which would double her wealth with probability 0.5, and leave her wealth unchanged with probability 0.5.

$A3$. If the price were exactly \$100 in each case, the transactions in (i) and (ii) would both be fair bets : they would leave the expected value of the person's wealth unchanged.

So a person who would want to undertake both actions must be risk averse at lower levels of wealth, and risk–loving at higher levels. A utility–of–wealth function $U(W)$ for such a person must have $U''(W) < 0$ for some levels of wealth below \$100, and $U''(W) < 0$ for some levels of wealth above \$100.

For example, suppose that

$$U(W) = 20000 \log W + W^2 \qquad (3-1)$$

Then

$$U''(W) = -\frac{20000}{W^2} + 2 \qquad (3-2)$$

so that $U''(W) < 0$ if and only if $W > 100$.

This person would get an expected utility level of $-\infty$ if some disaster could reduce her wealth W to 0 with positive probability. So she would certainly buy insurance against such a disaster at any price less than her whole wealth $W = 200$. If the insurnce were ctuarially fair, then buying full insurance would leave her with certain wealth of \$100. Buying the insurance gives her an expected utility of

$$U^I = 20000 \log 100 + (100)^2 \approx 102103 \qquad (3-3)$$

and going without insurance would leave her with expected utility

$$U^{NI} = (0.5)[20000 \log 200 + (200)^2] + (0.5)[20000 \log 0 + (0)^2] = -\infty \qquad (3-4)$$

so she would buy the insurance in case (i).

In case (ii), her expected utility if she did not undertake the investment would be

$$U^{NRI} = 20000 \log 200 + (200)^2 \approx 145966 \qquad (3-5)$$

If she undertook the investment, and had to pay a price of \$100, then she would have 100 if the investment failed, and $2(200) - 100 = 300$ if the investment succeeded, so that her expected utility would be

$$U^{RI} = (0.5)[20000 \log 100 + (100)^2] + (0.5)[20000 \log 300 + (300)^2 \approx 153089 \qquad (3-6)$$

so that $U^{RI} > U^{NRI}$.

$Q4$. (This is an example of the "St. Petersburg paradox".) If a person were a von Neumann–Morgenstern expected utility maximizer, with a constant coefficient of relative risk aversion of 2, what would be the certainty equivalent to the following compound lottery?

A coin is tossed once. If it lands "heads", she gets \$1000000. If it lands "tails", the coin is tossed again. If it lands "heads" on the second toss (after "tails" on the first), she gets \$2000000.

It it lands "tails" on both of the first two tosses, the coin is tossed again, and, if it lands "heads" on the third toss (after landing "tails" twice) she gets \$4000000. The coin–tossing continues until the first "heads", and her payoff will be 2^t million dollars, where t is the number of times that the coin landed "tails" consecutively before the first "heads".

*A*4. With a CRR expected utility, her utility from a wealth level of W can be written $U(W) = \frac{U^{1-\beta}}{1-\beta}$, where β is the coefficient of relative risk aversion. With $\beta = 2$, this means that the utility of a wealth level W is $-1/W$.

So if we measure wealth in millions of dollars, her expected utility from the gamble would be

$$EU = -(0.5)\frac{1}{1} - (0.25)\frac{1}{2} - (0.125)\frac{1}{4} - \cdots \qquad (4-1)$$

since the probability of a head on the first toss is 0.5, the probability that the first head is on the second toss is 0.25, and so on. Expression $(4-1)$ can be written

$$EU = -\frac{1}{2} - \frac{1}{8} - \frac{1}{32} - \cdots \qquad (4-2)$$

or

$$EU = -\frac{1}{2}\sum_{t=0}^{\infty}\frac{1}{4}^t \qquad (4-3)$$

Now a general rule about infinite sums is that

$$\sum_{t=0}^{\infty} x^t = \frac{1}{1-x} \qquad (4-4)$$

when $x < 1$.

[Proof : $(1-x)[\sum_{t=0}^{\infty} x^t] = (1-x)[1+x+x^2+\cdots] = (1+x+x^2+\cdots)-(x+x^2+x^3+\cdots) = 1$]

So, using this general rule,

$$EU = -\frac{1}{2}\frac{1}{1-0.25} = -\frac{2}{3} \qquad (4-5)$$

The certainty equivalent to this gamble is a certain amount of wealth CE such that $U(CE)$ equals the expected utility of the gamble, or $U(CE) = -\frac{2}{3}$. So

$$-\frac{1}{CE} = -\frac{2}{3} \qquad (4-6)$$

meaning that the certainty equivalent CE to the gamble is \$1.5 million.

*Q*5. If a person were a von Neumann–Morgenstern expected utility maximizer, with a constant coefficient of relative risk aversion of β, with wealth W_0 and she faced a 50% chance of losing all her wealth in some accident, for what values of β and W_0 would she be willing to buy an insurance policy which provided full insurance against the accident, at a cost of a fraction $\alpha > 0.5$ of her wealth?

$A5$. If she did not buy the insurance, then her expected utility would be

$$EU_0 = \frac{1}{1-\beta}[(0.5)(W_0)^{1-\beta} + (0.5)(0)^{1-\beta}] \qquad (5-1)$$

whereas if she paid αW_0 for the insurance, her utility would be a certain

$$EU_1 = \frac{1}{1-\beta}[(1-\alpha)W_0]^{1-\beta} \qquad (5-2)$$

First, note that the expression $0^{1-\beta}$ is infinite if $\beta > 1$. If her coefficient of relative risk aversion is greater than 1, then she would be willing to pay anything for insurance against the loss of all of her wealth (whatever the probability of loss).

Second, notice that when $\beta < 1$, expression $(5-2)$ is bigger than expression $(5-1)$ if and only if

$$[(1-\alpha)W_0]^{1-\beta} > (0.5)(W_0)^{1-\beta} \qquad (5-3)$$

So if $\beta < 1$, her choice of whether to buy the insurance does not depend on her initial wealth W_0 ; she will purchase the insurance if and only if

$$(1-\alpha)^{1-\beta} > 0.5 \qquad (5-4)$$

Taking logarithms of both sides of $(5-4)$, it will hold if and only if

$$(1-\beta)\log(1-\alpha) > \log(0.5) = -\log 2 \qquad (5-5)$$

or

$$\beta > 1 - \frac{\log(0.5)}{\log(1-\alpha)} \qquad (5-6)$$