*Q*1. Does the following production function exhibit decreasing, constant, or increasing returns to scale? Explain.

$$f(x_1, x_2, x_3) = 1 + x_1 \log(x_2 + 1) - \frac{1}{x_3 + 1}$$

*A*1. The marginal products of the three inputs are

$$f_1(x_1, x_2, x_3) = \log(x_2 + 1) \qquad (1-1)$$

$$f_2(x_1, x_2, x_3) = \frac{x_1}{x_2 + 1} \qquad (1-2)$$

$$f_3(x_1, x_2, x_3) = \frac{1}{(1 + x_3)^2} \qquad (1-3)$$

The “local” measure of scale economies for a production function is $\mu(x_1, x_2, x_3) \equiv \sum_i \mu_i(x_1, x_2, x_3)$ where

$$\mu_i(x_1, x_2, x_3) \equiv \frac{f_i(x_1, x_2, x_3)x_i}{f(x_1, x_2, x_3)} \quad i = 1, 2, 3 \qquad (1-4)$$

So here

$$\mu_1(x_1, x_2, x_3)f(x_1, x_2, x_3) = x_1 \log(x_2 + 1) \qquad (1-5)$$

$$\mu_2(x_1, x_2, x_3)f(x_1, x_2, x_3) = \frac{x_1 x_2}{x_2 + 1} \qquad (1-6)$$

$$\mu_3(x_1, x_2, x_3)f(x_1, x_2, x_3) = \frac{x_3}{(1 + x_3)^2} \qquad (1-7)$$

A production exhibits increasing returns to scale locally if and only if $\mu(x_1, x_2, x_3) > 1$. Here

$$[\mu(x_1, x_2, x_3) - 1]f(x_1, x_2, x_3) = \frac{x_1 x_2}{x_2 + 1} - \frac{(x_3)^2}{(1 + x_3)^2} \qquad (1-8)$$

The function exhibits increasing (decreasing) returns to scale if and only if expression $(1-8)$ is positive (negative). But depending on the values of x_1, x_2 and x_3, expression $(1-8)$ can be positive, negative, or 0.

For example, if $x_1 = 10, x_2 = 1, x_3 = 1$, then expression $(1-8)$ equals $4.75 > 0$, so that the production function exhibits irs locally : increasing (x_1, x_2, x_3) to (ax_1, ax_2, ax_3) where a is close to, but greater than, 1, will increase output by a factor greater than a. But if $x_1 = 0.5, x_2 = 0.5, x_3 = 1$, then expression $(1-8)$ equals $-0.0833 < 0$, so that the production function exhibits drs. (And, for example, if $x_1 = 1, x_2 = 0.333, x_3 = 1$, then $\mu(x_1, x_2, x_3) = 1$, so that the function exhibits locally crs at the point $(x_1, x_2, x_3) = (1, 0.333, 1)$ [and at many other points]).

*Q*2. Find the cost function $C(w_1, w_2, y)$ for the production function

$$f(x_1, x_2) = 2 - \frac{1}{x_1 + 1} - \frac{1}{x_2 + 1}$$

*A*2. The first–order conditions for cost minimization, that $\mu f_i(x_1, x_2) = w_i$ (for $i = 1, 2$) can here be written

$$\mu \frac{1}{(x_1 + 1)^2} = w_1 \tag{2 – 1}$$

$$\mu \frac{1}{(x_2 + 1)^2} = w_2 \tag{2 – 2}$$

so that

$$[\frac{(1 + x_1)}{(1 + x_2)}]^2 = \frac{w_2}{w_1} \tag{2 – 3}$$

which implies that

$$x_2 = \sqrt{\frac{w_1}{w_2}}(1 + x_1) - 1 \tag{2 – 4}$$

and

$$\frac{1}{1 + x_2} = \sqrt{\frac{w_2}{w_1}} \frac{1}{x_1} \tag{2 – 5}$$

which means that the quantity $y = f(x_1, x_2)$ of output can be written

$$y = 2 - \frac{1}{1 + x_1} - \sqrt{\frac{w_2}{w_1}} \frac{1}{1 + x_1} \tag{2 – 6}$$

or

$$y = 2 - \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_1}} \frac{1}{1 + x_1} \tag{2 – 7}$$

which can be re–arranged into the conditional input demand function for x_1,

$$x_1(w_1, w_2, y) = \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_1}} \frac{1}{2 - y} - 1 \tag{2 – 8}$$

and (substituting from (2 – 4))

$$x_2(w_1, w_2, y) = \frac{\sqrt{w_1} + \sqrt{w_2}}{\sqrt{w_2}} \frac{1}{2 - y} - 1 \tag{2 – 9}$$

The cost function $C(w_1, w_2, y)$ is the cost of the inputs, or

$$C(w_1, w_2, y) = w_1 x_1(w_1, w_2, y) + w_2 x_2(w_1, w_2, y) = \frac{[\sqrt{w_1} + \sqrt{w_2}]^2}{2 - y} - w_1 - w_2 \tag{2 – 10}$$

which also could be written

$$C(w_1, w_2, y) = \frac{(w_1 + w_2)(y - 1) + 2\sqrt{w_1 w_2}}{2 - y} \tag{2 – 11}$$

*Q*3. Find the cost function $C(w_1, w_2, w_3, y)$ for the production function

$$f(x_1, x_2, x_3) = \min(x_1, x_2) + x_3$$

*A*3. Because the production function combines features of perfect complements and of perfect substitutes, first–order conditions will not be sufficient here.

First, the "perfect complements" feature of the production function implies that $x_1(\mathbf{w}, y) = x_2(\mathbf{w}, y)$ for any input prices $\mathbf{w}$ and any output level y. The reason? Increasing x_1 above x_2 will cost the firm money (if $w_2 > 0$) but will not yield any more output : the marginal product of input 2 is 0 whenever $x_2 > x_1$. Similarly, $MP_1 = 0$ if $x_1 > x_2$.

Now the "perfect substitutes" feature of the production function means that the firm should never use any of input #3 if $w_3 > w_1 + w_2$: increasing **each** of x_1 and x_2 by ϵ while reducing x_3 by ϵ will not change the level of output produced, but will lower costs by $[w_3 - w_1 - w_2]\epsilon$. Similarly, it will never be optimal to choose $x_1 > 0$ or $x_2 > 0$ if $w_1 + w_2 > w_3$.

So the conditional input demands are

$$x_1(\mathbf{w}, y) = x_2(\mathbf{w}, y) = y \quad ; \quad x_3(\mathbf{w}, y) = 0 \quad \text{if} \quad w_1 + w_2 < w_3 \tag{3-1}$$

$$x_1(\mathbf{w}, y) = x_2(\mathbf{w}, y) = 0 \quad ; \quad x_3(\mathbf{w}, y) = y \quad \text{if} \quad w_1 + w_2 > w_3 \tag{3-2}$$

and the conditional input demands are undefined when $w_1 + w_2 = w_3$ (except that it must be true that $x_1(\mathbf{w}, y) = x_2(\mathbf{w}, y) = y - x_3(\mathbf{w}, y)$).

From equations $(3-1)$ and $(3-2)$, the cost function is

$$C(\mathbf{w}, y) = \min[(w_1 + w_2)y, w_3 y] \tag{3-3}$$

*Q*4. Find the profit function $\pi(p, w_1, w_2)$ for a firm with a production function

$$f(x_1, x_2) = \sqrt{\min(x_1, x_2)}$$

*A*4. (As the solution to problem #3 suggests), in this case, the fact that the two inputs are perfect complements means that the conditional input demands must obey

$$x_1(w_1, w_2, y) = x_2(w_1, w_2, y) \tag{4-1}$$

so that

$$x_1(w_1, w_2, y) = y^2 = x_2(w_1, w_2, y) \tag{4-2}$$

meaning that

$$C(w_1, w_2, y) = (w_1 + w_2)y^2 \tag{4-3}$$

One way of solving the firm's profit maximization problem is to choose an output level y so as to maximize

$$py - C(w_1, w_2, y) \qquad (4-4)$$

In this example, that means the maximization of

$$py - (w_1 + w_2)y^2 \qquad (4-5)$$

with respect to y. The first–order condition for this maximization is

$$y = \frac{p}{2(w_1 + w_2)} \qquad (4-6)$$

Substituting from $(4-6)$ and $(4-3)$ into $(4-4)$ yields

$$\pi(p, w_1, w_2) = \frac{1}{4}\frac{p^2}{w_1 + w_2} \qquad (4-7)$$

*Q*5. What is the equation of the long–run supply curve for a perfectly–competitive industry, in which each of the (many) identical firms has a long run total cost function

$$TC(q) = q^3 - 24q^2 + 200q$$

where q is the quantity of output produced by the firm?

*A*5. Since the long–run total cost function is

$$TC(q) = q^3 - 24q^2 + 200q$$

then a firm's long–run marginal cost and average cost functions are

$$MC(q) = TC'(q) = 3q^2 - 48q + 200 \qquad (2-1)$$

$$AC(q) = \frac{TC(q)}{q} = q^2 - 24q + 200 \qquad (2-2)$$

Differentiating yet again,

$$MC'(q) = 6q - 48 \qquad (2-3)$$

$$AC'(q) = 2q - 24 \qquad (2-4)$$

From equations $(2-3)$ and $(2-4)$ both the marginal and average cost curves are U–shaped, with minima at $q = 8$ and $q = 12$ respectively. When $q = 12$,

$$MC(q) = 3(12^2) - 48(12) + 200 = 56 \qquad (2-5)$$

$$AC(q) = 144 - 24(12) + 200 = 56 \qquad (2-6)$$

confirming that the AC and MC curves cross at the bottom of the AC curve.

With identical firms in perfect competition, in the long run it must be the case that $p = MC$ if firms each maximize their profits, and that $p = AC$ if there is free entry and exit. The only quantity q for which $MC = AC$ is the bottom of each firm's AC curve, $q = 12$.

Thus, in the long–run, the price must be 56, and each firm in the industry must produce 12 units of output. The industry long–run curve is horizontal, at a height of $p = 56$.