Revenue Equivalence

1 First Price

The person's value is v.

Every other person uses the bidding rule $b(w)$, where w is the other bidder's valuation.

So the person of value v can be viewed as picking a type w to mimic : that is, for the person of type v, choosing a bid β is equivalent to choosing to act like a person of type w, where $\beta = b(w)$.

The payoff to the person of type v, if she chooses to act like a person of type w, is

$$[F(w)]^{n-1}(v - b(w)) \tag{1}$$

where $F(\cdot)$ is the distribution function for people's valuations : she will win the object only if all $n-1$ other people have a value less than w.

Choosing a type–to–mimic w so as to maximize expression (1) leads to the first–order condition

$$(n-1)[F(w)]^{n-2}f(w)(v - b(w)) - [F(w)]^{n-1}b'(w) = 0 \tag{2}$$

which can also be written

$$\frac{d}{dw}[\Phi(w)] = (n-1)[F(w)]^{n-2}f(w)v \tag{3}$$

where

$$\Phi(w) \equiv [F(w)]^{n-1}b(w) \tag{4}$$

In a symmetric Nash equilibrium, every bidder chooses the same rule : so that a bidder of type v should bid $b(v)$ in equilibrium.

That means that the person of value v will choose to act like a person of type v ; that is, the w which solves equation (2) or (3) is $w = v$, which means that equation (3) becomes

$$\frac{d}{dv}[\Phi(v)] = (n-1)[F(v)]^{n-2}f(v)v \tag{5}$$

Equation (5) is a differential equation defining the function $\Phi(v)$. Integrating both sides — and using the fact that $\Phi(0) = 0$ — gives the solution

$$\Phi(v) = \int_0^v (n-1)[F(x)]^{n-2}f(x)x\,dx \tag{6}$$

Since

$$b(v) \equiv \frac{\Phi(v)}{[F(v)]^{n-1}}$$

equation (6) implies that

$$b(v) = (n-1)\frac{\int_0^v [F(x)]^{n-2} x f(x)}{[F(v)]^{n-1}} \tag{7}$$

or

$$b(v) = \frac{\int_0^v x d[F(x)]^{n-1}}{[F(v)]^{n-1}} \tag{8}$$

2 Second Price

In a second price auction, every person has a (weakly) dominant strategy, which is to bid her true valuation.

What is the expected revenue the auctioneer will collect from a type–v bidder, in a second price auction, **conditional** on the person winning the auction?

Let $G(\cdot)$ be the distribution function for the **highest** of the other $n-1$ bids. The probability that the person of valuation v wins the object, in an n–person second–price auction is therefore $G(v)$, the probability that the other $n-1$ bidders value the object less than she does. The price she will pay, should she be the high bidder, is the highest of the $n-1$ other bids. So the expected revenue collected from the bidder of type v, conditional on her winning the auction, is

$$r(v) \equiv \frac{\int_0^v x dG(x)}{G(v)} \tag{9}$$

Now the probability that the highest of the $n-1$ other bids is less than v is $[F(v)]^{n-1}$. So

$$G(v) = [F(v)]^{n-1} \tag{10}$$

So equation (9) can be written

$$r(v) = \frac{\int_0^v x d[F(x)]^{n-1}}{[F(v)]^{n-1}} \tag{11}$$

Comparing equations (8) and (11), we have

$$r(v) = b(v) \tag{12}$$

3 Revenue Equivalence

Suppose that some person values the object at v. If the object is auctioned off, how much will she pay?

If it's a first–price auction, she will bid $b(v)$, which is defined in expression (8). That's her bid ; she may or may not win the object. But if she does win the object, $b(v)$ is what she will pay, since that's what she bid.

If it's a second–price, she will bid her actual value v. Again, she may or may not win the object with this bid. The probability she does win is $G(v)$. And with a second–price auction, the amount that she actually pays depends on what the second–highest bid. The expected value of what she will wind up paying, if she does win the object, was defined by expression (11).

Therefore, the expected value of what a person with value v pays, should she win the object, is the same in equilibrium, whether the auction is a first–price or a second–price auction.

That means that the expected revenue collected by the auctioneer will be the same in either auction : in either case, the expected revenue can be written

$$ER = \int_0^\infty r(v) dH(v) \tag{13}$$

where $H(v)$ is the probability that the highest of the n values is v or less – so that

$$H(v) = [F(v)]^n \tag{14}$$

4 The Revelation Principle

Suppose that the auctioneer decides to skip all the formality of the auction.

Since the highest–value person wins the object in the first– or second–price auction, why not simply ask people what their values are, and award the object to the person who announces the highest value?

To implement this mechanism, the mechanism must have rules which induce people to tell the truth.

So consider a mechanism in which the organizer asks each person to report her valuation, and promises to award the object to the person who announces the highest value, in exchange for a fee $P(v)$ which depends on the value the winner has announced.

Consider the incentives faced by a bidder of value v, if this bidder thinks that everyone else (except for her) has announced his value truthfully. This bidder, if she announces that her own value is w – which may or may not be the truth — will get the object with probability $[F(w)]^{n-1}$. So her expected payoff, if she announces her value as w, will be

$$[F(w)]^{n-1}(v - P(w)) \tag{15}$$

She will choose the announced value w to maximize her expected payoff (15).

But notice that expression (15) is exactly the same as expression (1), except that $b(w)$ is now called $P(w)$. In order to get the person to choose to tell the truth ($v = w$), the derivative of expression (15) with respect to w must equal 0 when $w = r$.

So the fee schedule $P(v)$ chosen by the organizer must be exactly the same as $b(v)$, if the organizer is to induce people to report their values truthfully.

That means that the expected revenue from this fee schedule must be the expected value of the bid $b(v)$ of the highest–value bidder, which was defined in expression (13).

So any "direct mechanism", in which we simply ask people their value for the object, must have expected revenue defined by (13), if the mechanism actually induces people to reveal truthful their values.

The two auctions considered here were indirect mechanisms : no–one asked a bidder directly what her value was. But these indirect mechanisms have the same efficiency property as the direct mechanism. No matter what the actual realization of people's values, the object goes to the person who values it most.

And it must be true that any indirect mechanism (such as an auction) will have the exact same revenue as the equivalent direct mechanism — here "equivalent" means that the direct and indirect mechanisms always allocate the object to the same person. So the general result is that **any** auction rules will yield the same expected revenue as this direct mechanism, provided that the auction is efficient. "Efficient" means that the auction always allocates the object to the person who values the object most highly.