Definitions

1. A function $f : \Re^n \rightarrow \Re$ is **quasi–concave** if, for any $\mathbf{x}$, the set

$$\succeq (\mathbf{x}) = \{\mathbf{y} \in \Re^n | f(\mathbf{y}) \geq f(\mathbf{x})\}$$

is a **convex set**.

2. A function $f : \Re^n \rightarrow \Re$ is **quasi–concave** if

$$f(t\mathbf{x}^1 + (1-t)\mathbf{x}^\mathbf{2}) \geq \min\left[\mathbf{f}(\mathbf{x}^\mathbf{1}), \mathbf{f}(\mathbf{x}^\mathbf{2})\right]$$

for any $\mathbf{x}^1$ and $\mathbf{x}^2$ in $\Re^n$, and any $0 < t < 1$

3. A function $f : \Re^n \rightarrow \Re$ is **quasi–concave** if

$$\mathbf{v}'\mathbf{H}(\mathbf{x})\mathbf{v} \leq 0$$

for any $\mathbf{x} \in \Re^n$ and any direction $\mathbf{v}$ such that

$$\nabla f(\mathbf{x}) \cdot \mathbf{v} = 0$$

where $\mathbf{H}(\mathbf{x})$ is the matrix of second derivatives of $f(\mathbf{x})$ and $\nabla f(\mathbf{x})$ is the vector of first derivatives of the function $f(\mathbf{x})$.

4.A function $f : \Re^n \rightarrow \Re$ is **quasi–concave** if the determinants of the 2–by–2, 3–by–3, $\ldots$, n–by–n matrices in the top left corner of the bordered Hessian matrix

$$\mathbf{H}^* \equiv \begin{pmatrix} 0 & f_1 & f_2 & \cdots & f_n \\ f_1 & f_{11} & f_{12} & \cdots & f_{1n} \\ f_2 & f_{21} & f_{22} & \cdots & f_{2n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ f_n & f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix}$$

alternate in sign.

That is, the function $f(\mathbf{x})$ is quasi–concave if, for every value of $\mathbf{x}$,

$$\begin{pmatrix} 0 & f_1 \\ f_1 & f_{11} \end{pmatrix}$$

has a negative determinant,

$$\begin{pmatrix} 0 & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}$$

has a positive determinant, and so on
(where f_i is the first derivative of $f(\mathbf{x})$ with respect to x_i, and f_{ij} is the second derivative of $f(\mathbf{x})$ with respect to x_i and x_j)