Constant Elasticity of Substitution [CES] Preferences

$$u(\mathbf{x}) = (x_1^\rho + x_2^\rho + \cdots + x_n^\rho)^{1/\rho}$$

where $-\infty < \rho < 1$ and $\rho \neq 0$

marginal utilities

$$\frac{\partial u}{\partial x_i} = \frac{1}{\rho}(x_1^\rho + x_2^\rho + \cdots + x_n^\rho)^{1/\rho - 1} \rho x_i^{\rho - 1}$$

first–order conditions for utility maximization

$$(x_1^\rho + x_2^\rho + \cdots + x_n^\rho)^{1/\rho - 1} x_i^{\rho - 1} - \lambda p_i = 0 \quad i = 1, 2, \cdots, n$$

along with the budget constraint

$$\sum_j p_j x_j = y$$

manipulating the equations

Take the first–order condition for consumption of commodity i, and divide both sides by the first–order condition for the consumption of commodity 1. What results is

$$(\frac{x_i}{x_1})^{\rho-1} = \frac{p_i}{p_1}$$

or

$$x_i = (\frac{p_i}{p_1})^{1/(\rho-1)} x_1 \qquad (1)$$

which implies that

$$\begin{aligned} p_i x_i &= p_i (p_i)^{1/(\rho-1)} p_1^{-1/(\rho-1)} x_1 \\ &= (p_i)^{\rho/(\rho-1)} (p_1)^{-1/(\rho-1)} x_1 \qquad (2) \\ &= p_i^r p_1^{1-r} x_1 \end{aligned}$$

Now let

$$r \equiv \frac{\rho}{\rho - 1}$$

Add up equation 2 over all n commodities to get

$$\sum_{j=1}^{n}(p_j x_j) = [\sum_{j=1}^{n} p_j^r](p_1)^{1-r} x_1 \qquad (3)$$

The budget constraint says that the left side of equation 3 is y, which means that

$$x_1 = \frac{p_1^{r-1} y}{\sum_{j=1}^{n} p_j^r}$$

which is the Marshallian demand function for commodity number 1. Substituting back into equation (1) shows that, for any commodity i,

$$x_i(\mathbf{p}, y) = \frac{p_i^{r-1} y}{\sum_{j=1}^{n} p_j^r}$$

defining the Marshallian demand functions when preferences are CES.