CES : Expenditure Function and Hicksian Demands

expenditure minimization

minimize $\mathbf{p} \cdot \mathbf{x}$ subject to

$$[\sum_{i=1}^{n} x_i^{\rho}]^{1/\rho} \geq u \tag{1}$$

so the Lagrangean is

$$\mathbf{p} \cdot \mathbf{x} + \mu[u - [\sum_{i=1}^{n} x_i^{\rho}]^{1/\rho}] \tag{2}$$

with first–order conditions

$$p_i = \mu[\sum_{k=1}^{n} x_k^{\rho}]^{1/\rho - 1} x_i^{\rho-1} \quad i = 1, 2, \ldots, n \tag{3}$$

re–arranging (3),

$$\frac{x_i}{x_j} = [\frac{p_i}{p_j}]^{1/(\rho-1)} \tag{4}$$

for any 2 goods i and j, so that, in particular

$$x_i = [\frac{p_i}{p_1}]^{1/(\rho-1)} x_1 \tag{5}$$

which means that

$$u = x_1 [\sum_{j=1}^{n} (\frac{p_j}{p_1})^{\rho/(\rho-1)}]^{1/\rho} \tag{6}$$

which, in turn, can be re–arranged to

$$x_1 = p_1^{-1/(1-\rho)} [\sum_{j=1}^{n} p_i^{\rho/(\rho-1)}]^{-1/\rho} u \tag{7}$$

which is a Hicksian demand function

since

$$r \equiv \frac{\rho}{\rho - 1}$$

so that

$$\rho = -\frac{r}{1 - r}$$

equation (7) can be written

$$x_1^h(\mathbf{p}, u) = p_1^{r-1}[\sum_{j=1}^{n} p_j^r]^{1/r-1} u \qquad (8)$$

and the Hicksian demand function for any other good i is

$$x_i^h(\mathbf{p}, u) = p_i^{r-1}[\sum_{j=1}^{n} p_j^r]^{1/r-1} u \qquad (9)$$

CES : Expenditure Function

the expenditure function is the sum of expenditure $p_i x_i^h(\mathbf{p}, u)$ on all the goods ; from equation (9),

$$\mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u) = \sum_{i=1}^{n} p_i(p_i^{r-1})[\sum_{j=1}^{n} p_j^r]^{1/r-1} u \qquad (10)$$

or

$$\mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u) = [\sum_{i=1}^{n} p_i^r][\sum_{i=1}^{n} p_i^r]^{1/r-1} \qquad (11)$$

meaning that the expenditure function for CES preferences is

$$e(\mathbf{p}, u) = [\sum_{i=1}^{n} p_i^r]^{1/r} u \qquad (12)$$