Properties of Hicksian Demand Functions

fact : the matrix of second derivatives of an expenditure function $e(\mathbf{p}, u)$ with respect to the prices is a **negative semi–definite** matrix

[proof? : $e(\mathbf{p}, u)$ is a concave function of the vector $\mathbf{p}$ of prices (concave, not just quasi–concave) — that's part of Theorem 1.7 in *Jehle and Reny*. Theorem $A.2.4$ in the appendix of *Jehle and Reny* shows that the matrix of second derivatives of a concave function must be negative semi–definite.]

fact : every negative semi–definite matrix must have non–positive numbers on the diagonal

implication : $e_{ii} \leq 0$ for every good i

Shepherd's Lemma : $e_i = x_i^H(\mathbf{p}, u)$

implication of Shepherd's Lemma

$$e_{ii} = \frac{\partial x_i^H}{\partial p_i} \leq 0 \tag{1}$$

Law of Demand : the compensated (Hicksian) demand for any good is a non–increasing function of the good's own price

Young's Theorem : if a function $f(\mathbf{z})$ is twice continuously differentiable, then $\frac{\partial^2 f}{\partial z_i \partial z_j} = \frac{\partial^2 f}{\partial z_j \partial z_i}$

so $e_{ij} = e_{ji}$; so that Shepherd's Lemma also implies that

$$\frac{\partial x_i^H}{\partial p_j} = \frac{\partial x_j^H}{\partial p_i} \tag{2}$$

Relations Among Marshallian Demands

budget constraint :

$$\sum_{i=1}^{n} p_i x_i^M(\mathbf{p}, y) = y \tag{3}$$

differentiate both sides of (3) with respect to y to get

$$\sum_{i=1}^{n} p_i \frac{\partial x_i^M}{\partial y} = 1 \tag{4}$$

which also can be written

$$\sum_{i=1}^{n} \frac{\partial x_i^M}{\partial y} \frac{y}{x_i} \frac{p_i x_i}{y} = 1 \tag{5}$$

define income elasticity of demand for good i

$$\eta_i \equiv \frac{\partial x_i}{\partial y}\frac{y}{x_i} \tag{6}$$

and share s_i of good i in the person's budget

$$s_i \equiv \frac{p_i x_i}{y} \tag{7}$$

and equation (5) becomes

$$\sum_{i=1}^{n} s_i \eta_i = 1 \tag{8}$$

differentiate both sides of (3) with respect to some price p_j,

$$x_j^M(\mathbf{p}, y) + \sum_{i=1}^{n} p_i \frac{\partial x_i^M}{\partial p_j} = 0 \tag{9}$$

which can also be written

$$\sum_{i=1} \frac{p_i x_i}{y} \frac{\partial x_i}{\partial p_j} \frac{p_j}{x_i} = -\frac{p_j x_j}{y} \tag{10}$$

if the cross–price elasticity ϵ_{ij} of Marshallian demand for good i with respect to the price of good j is defined

$$\epsilon_{ij} \equiv \frac{\partial x_i^M(\mathbf{p}, y)}{\partial p_j} \frac{p_j}{x_i} \tag{11}$$

then equation (10) can be written

$$\sum_{i=1}^{n} s_i \epsilon_{ij} = -s_j \tag{12}$$