GS/ECON 5010 APPLIED MICROECONOMICS

Midterm Exam Answers to Midterm October 2004

Q1. Derive a consumer's Marshallian (uncompensated) and Hicksian (compensated) demand functions for all three commodities, if her preferences can be represented by the utility function

$$u(x_1, x_2, x_3) = x_1 + \ln x_2 + \ln x_3$$

(You may assume that her income is greater than $2p_1$.)

A1. The consumer's Marshallian demand function is the solution to her utility maximization problem, maximization of $u(x_1, x_2, x_3)$ subject to $p_1x_1+p_2x_2+p_3x_3 \leq y$. Setting up the Lagrangean

$$\mathcal{L}(x_1, x_2, x_3; \lambda) = x_1 + \ln x_2 + \ln x_3 + \lambda(y - p_1x_1 - p_2x_2 - p_3x_3)$$

yields first–order conditions

$$1 = \lambda p_1 \qquad (1-1)$$

$$\frac{1}{x_2} = \lambda p_2 \qquad (1-2)$$

$$\frac{1}{x_3} = \lambda p_3 \qquad (1-3)$$

So that

$$x_2 = \frac{p_1}{p_2} \qquad (1-4)$$

$$x_3 = \frac{p_1}{p_3} \qquad (1-5)$$

Equations $(1-4)$ and $(1-5)$ define immediately the consumer's Marshallian demand functions for goods 1 and 2, since they express quantity demand as a function of prices and income. To find the Marshallian demand for good 1, substitute from the budget constraint,

$$x_1 = \frac{y - p_2x_2 - p_3x_3}{p_1}$$

which (from equations $(1-4)$ and $(1-5)$) implies that

$$x_1^M(\mathbf{p}, y) = \frac{y}{p_1} - 2 \qquad (1-6)$$

[This equation makes sense only if $y \geq 2p_1$, as assumed in the question. If $y < 2p_1$, then the consumer would choose $x_1 = 0$, $x_2 = \frac{y}{2p_2}$, and $x_3 = \frac{y}{2p_3}$.]

Since quantity demand of goods 2 and 3 do not depend on income, then the Hicksian demands are just the Marshallian demands

$$x_i^H(\mathbf{p}, u) = x^M(\mathbf{p}, y) = p\frac{p_1}{p_i} \quad i = 2, 3$$

The easiest way to find the Hicksian demand function for good 1 is probably to substitute from the utility function :

$$x_1^H(\mathbf{p}, u) + \ln[x_2^H(\mathbf{p}, u)] + \ln[x_3^H(\mathbf{p}, u)] = u$$

which implies that

$$x_1^H(\mathbf{p}, u) = u - \ln\frac{p_1}{p_2} - \ln\frac{p_1}{p_3}$$

or

$$x_1^H(\mathbf{p}, u) = u - 2\ln p_1 + \ln p_2 + \ln p_3$$

[Alternatively, you could have calculated the indirect utility function from the Marshallian demands :

$$v(\mathbf{p}, y) = \frac{y}{p_1} + 2\ln p_1 - \ln p_2 - \ln p_3$$

and used the fact that $e(\mathbf{p}, v(\mathbf{p}, y)) = y$ to get

$$e(\mathbf{p}, u) = up_1 - 2p_1\ln p_1 + p_1\ln p_2 + p_1\ln p_3 + 2p_1 \qquad (1-6)$$

and then used Shephard's Lemma to get the Hicksian demand for good 1.]

Q2. Derive the relationship ("Engel aggregation") among a consumer's income elasticities of demand for the commodities which she consumes.

A2. Engel aggregation, defined on page 58 in the text, refers to the condition

$$\sum_{i=1}^{n} s_i\eta_i = 1$$

where s_i is the proportion of her income that the consumer spends on good i, and η_i is the income elasticity of demand for good i.

So the condition says that an appropriately–weighted sum of income elasticities of demand for all goods must equal 1.

To derive it, differentiate the budget constraint

$$\sum_{i=1}^{n} p_i x_i^M(\mathbf{p}, y) = y$$

with respect to y, to get

$$\sum_{i=1}^{n} p_i\frac{\partial x_i}{\partial y} = 1$$

If each term on the left side of the above equation is multiplied and divided by x_i/y, it becomes

$$\sum_{i=1}^{n} \frac{p_i x_i}{y} \frac{\partial x_i}{\partial y} \frac{y}{x_i} = 1$$

which is the Engel aggregation result, since

$$s_i \equiv \frac{p_i x_i}{y}$$

and

$$\eta_i \equiv \frac{\partial x_i}{\partial y} \frac{y}{x_i}$$

Q3. If person 1's utility–of–wealth function can be written $g[u(W)]$, where $u(W)$ is person 2's utility–of–wealth function, and $g(\cdot)$ is an increasing, concave function, which person is more risk averse? What is the relationship between the two people's coefficients of absolute risk aversion?

A3. Person 1 will be more risk averse (if $g(\cdot)$ is strictly concave). That is, a "concavification" of a utility–of–wealth function makes the person more risk averse, as discussed on page 109 of the textbook.

The proof in the text uses Jensen's Inequality, that $g(\sum_i a_i x_i) > \sum_i a_i g(x_i)$ if $g(\cdot)$ is strictly concave, where the a_i's are non–negative weights which sum to 1. The application of Jensen's inequality : consider some gamble $(p_1 \circ x_1, p_2 \circ w_2, \ldots, p_n \circ x_n)$. Let CE_2 be person 2's certainty equivalent to the gamble, so that

$$\sum_i p_i u(x_i) = u(CE_2)$$

Jensen's inequality says that

$$\sum_i p_i g(u(x_i)) < g(u(CE_2)) \tag{3 − 1}$$

But the left side of inequality $(3-1)$ is $g(u(CE_1))$, where CE_1 is person 1's certainty equivalent to the gamble. Since $g(\cdot)$ and $u(\cdot)$ are increasing functions, then $(3-1)$ implies that $CE_1 < CE_2$. Person 1 has a lower certainty equivalent for any gamble than person 2.

More directly, calculate the two people's coefficients of absolute risk aversion, $R_a^1(W)$ and $R_a^2(W)$. Let $v(W) \equiv g(u(W))$ be person 1's utility–of–wealth function. Then

$$v'(W) = g'(u(W))u'(W) \tag{3 − 2}$$

so that

$$v''(W) = g''(u(W))[u'(W)]^2 + g'(u(W))u''(W) \tag{3 − 3}$$

Then

$$R_a^1(W) = -\frac{v''(W)}{v'(W)} = -\frac{g''(u(W))[u'(W)]^2 + g'(u(W))u''(W)}{g'(u(W))u'(W)} = -[u'(W)\frac{g''(u(W))}{g'(u(W))} + \frac{u''(W)}{u'(W)}] \tag{3 − 4}$$

Equation $(3-4)$ says that

$$R_a^1(W) = R_a^2(W) + \frac{-u'g''}{g'} > R_a^2(W) \tag{3-5}$$

Equation$(3-5)$ shows that person 1 has a higher coefficient of risk aversion than person 2, proving that she is more risk averse. It also provides the relationship between the two people's coefficients of absolute risk aversion.

Q4. Derive the cost function for a firm with a production function

$$f(x_1, x_2) = \ln x_1 + \ln x_2$$

A4. The cost function results from the firm minimizing the cost of producing a given level y of output, the solution to the problem of minimizing $w_1x_1 + w_2x_2$ subject to $f(x-1, x_2) = y$. This implies a Lagrangean

$$\mathcal{L}(x_1, x_2; \lambda) = w_1x_1 + w_2x_2 + \lambda(y - \ln x_1 - \ln x_2)$$

The first–order conditions for the minimization are

$$w_1 = \frac{\lambda}{x_1} \tag{4-1}$$

$$w_2 = \frac{\lambda}{x_2} \tag{4-2}$$

implying that

$$x_2 = \frac{w_1}{w_2}x_1 \tag{4-3}$$

Substituting $(4-3)$ into the constraint that $y = f(x_1, x_2)$,

$$\ln x_1 + \ln\,[\frac{w_1}{w_2}x_1] = y \tag{4-4}$$

Since $\ln(ab) = \ln a + \ln b$ then equation $(4-4)$ implies that

$$y = 2\ln x_1 + \ln w_1 - \ln w_2 \tag{4-5}$$

Or

$$\ln x_1 = \frac{y - \ln w_1 + \ln w_2}{2} \tag{4-6}$$

Taking the exponent of both sides of $(4-6)$,

$$x_1 = \sqrt{e^y \frac{w_2}{w_1}} \tag{4-7}$$

and (using $(4-3)$)

$$x_2 = \sqrt{e^y \frac{w_1}{w_2}} \tag{4-8}$$

Equations $(4-7)$ and $(4-8)$ are the firm's conditional input demand functions. The cost function $C(\mathbf{w}, y)$ is the cost of these inputs, $w_1x_1(\mathbf{w}, y) + w_2x_2(\mathbf{w}, y)$, so that

$$C(\mathbf{w}, y) = w_1\sqrt{e^y\frac{w_2}{w_1}} + w_2\sqrt{e^y\frac{w_1}{w_2}}$$

or

$$C(\mathbf{w}, y) = e^y\sqrt{w_1w_2} \tag{4-9}$$